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A quantum duality principle for coisotropic subgroups and Poisson quotients

Nicola Ciccoli[a], Fabio Gavarini[b,*]

[a]Dipartimento di Matematica, Università di Perugia, Via Vanvitelli 1, I-06123 Perugia, Italy
[b]Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", Via della Ricerca Scientifica 1, I-00133 Roma, Italy

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Abstract

We develop a quantum duality principle for coisotropic subgroups of a (formal) Poisson group and its dual: namely, starting from a quantum coisotropic subgroup (for a quantization of a given Poisson group) we provide functorial recipes to produce quantizations of the dual coisotropic subgroup (in the dual formal Poisson group). By the natural link between subgroups and homogeneous spaces, we argue a quantum duality principle for Poisson homogeneous spaces which are Poisson quotients, i.e. have at least one zero-dimensional symplectic leaf. As an application, we provide an explicit quantization of the homogeneous SL_n^*-space of Stokes matrices, with the Poisson structure given by Dubrovin and Ugaglia.

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0. Introduction

The natural semiclassical counterpart of the study of quantum groups is the theory of Poisson groups: indeed, Drinfeld himself introduced Poisson groups as the semiclassical

* Corresponding author.
E-mail addresses: ciccoli@dipmat.unipg.it (N. Ciccoli), gavarini@mat.uniroma2.it (F. Gavarini).

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limits of quantum groups. Therefore, it should be no surprise to anyone, anymore, that the geometry of quantum groups gain in clarity and comprehension when its connection with Poisson geometry is more transparent. The same can be observed when referring to homogeneous spaces.

In fact, in the study of Poisson homogeneous spaces, a special rôle is played by *Poisson quotients*. These are those Poisson homogeneous spaces whose symplectic foliation has at least one zero-dimensional leaf, so they can be thought of as pointed Poisson homogeneous spaces, just like Poisson groups themselves are pointed by the identity element. When looking at quantizations of a Poisson homogeneous space, one finds that the existence is guaranteed only if the space is a quotient (cf. [EK2]). Thus the notion of Poisson quotient shows up naturally also from the point of view of quantization (see [Ci]).

Poisson quotients are a natural subclass of Poisson homogeneous G-spaces (G a Poisson group), best adapted to the usual relation between homogeneous G-spaces and subgroups of G: they correspond to *coisotropic* subgroups. The quantization process for a Poisson G-quotient then corresponds to a like procedure for the attached coisotropic subgroup of G. Also, when following an infinitesimal approach one deals with Lie subalgebras of the Lie algebra $\mathfrak{g}$ of G, and the coisotropy condition has its natural counterpart in this Lie algebra setting; the quantization process then is to be carried on for the Lie subalgebra corresponding to the initial homogeneous G-space.

When quantizing Poisson groups (or Lie bialgebras), a precious tool is the quantum duality principle (QDP). Loosely speaking this guarantees that any quantized enveloping algebra can be turned (roughly speaking) into a quantum function algebra for the dual Poisson group; viceversa any quantum function algebra can be turned into a quantization of the enveloping algebra of the dual Lie bialgebra. More precisely, let $\mathcal{QUEA}$ and $\mathcal{QFSHA}$ respectively be the category of all quantized universal enveloping algebras (QUEA) and the category of all quantized formal series Hopf algebras (QFSHA), in Drinfeld's sense. After its formulation by Drinfeld (see [Dr1, §7]) the QDP establishes a category equivalence between $\mathcal{QUEA}$ and $\mathcal{QFSHA}$ via two functors, $(\ )' \colon \mathcal{QUEA} \longrightarrow \mathcal{QFSHA}$ and $(\ )^\vee \colon \mathcal{QFSHA} \longrightarrow \mathcal{QUEA}$, such that, starting from a QUEA over a Lie bialgebra (resp. from a QFSHA over a Poisson group) the functor $(\ )'$ (resp. $(\ )^\vee$) gives a QFSHA (resp. a QUEA) over the dual Poisson group (resp. the dual Lie bialgebra). In a nutshell, $U_\hbar(\mathfrak{g})' = F_\hbar[[G^*]]$ and $F_\hbar[[G]]^\vee = U_\hbar(\mathfrak{g}^*)$ for any Lie bialgebra $\mathfrak{g}$. So from a quantization of any Poisson group this principle gets out a quantization of the dual Poisson group too.

In this paper, we establish a similar QDP for (closed) coisotropic subgroups of a Poisson group G, or equivalently for Poisson G-quotients, sticking to the formal approach which is best suited for dealing with quantum groups à la Drinfeld. Namely, given a Poisson group G assume quantizations $U_\hbar(\mathfrak{g})$ and $F_\hbar[[G]]$ of it are given; then any formal coisotropic subgroup K of G has two possible algebraic descriptions via objects related to $U(\mathfrak{g})$ or $F[[G]]$, and similarly for the formal Poisson quotient G/K. Thus the datum of K or equivalently of G/K is described algebraically in four possible ways: by quantization of such a datum we mean a quantization of any one of these four objects. Our "QDP" now is a series of functorial recipes to produce, out of a quantization of K or G/K as before, a similar quantization of the so-called *complementary dual* of K, i.e. the coisotropic subgroup $K^\perp$ of G^* whose tangent Lie bialgebra is just $\mathfrak{k}^\perp$ inside $\mathfrak{g}^*$, or of the associated Poisson G^*-quotient, namely $G^*/K^\perp$.

We would better stress that, just like the QDP for quantum groups, ours is by no means an existence result: instead, it can be thought of as a *duplication result*, in that it yields a new quantization (for a complementary dual object) out of one given from scratch.

As an aside remark, let us comment on the fact that the more general problem of quantizing coisotropic manifolds of a given Poisson manifold, in the context of deformation quantization, has recently raised quite some interest (see [BGHHW,CF]).

As an example, in the last section we show how we can use this quantum duality principle to derive new quantizations from known ones. The example is given by the Poisson structure introduced on the space of Stokes matrices by Dubrovin (see [Du]) and Ugaglia (see [Ug]) in the framework of moduli spaces of semisimple Frobenius manifolds. It was Boalch (cf. [Bo]) that first gave an interpretation of Dubrovin–Ugaglia brackets in terms of Poisson–Lie groups. We will rather follow later work by Xu (see [Xu]) where it was shown how Boalch construction may be equivalently interpreted as quotient Poisson structure of the dual Poisson–Lie group G^* of the standard $SL_n(\Bbbk)$. In more detail the Poisson space of Stokes matrices $G^*/H^\perp$ is the dual Poisson space to the Poisson space $SL_n(\Bbbk)/SO_n(\Bbbk)$. It has to be noted that the embedding of $SO_n(\Bbbk)$ in $SL_n(\Bbbk)$ is known to be coisotropic but not Poisson. Starting, then, from results obtained by Noumi in [No] related to a quantum version of the embedding $SO_n(\Bbbk) \hookrightarrow SL_n(\Bbbk)$ we are able to interpret them as an explicit quantization of the Dubrovin–Ugaglia structure. We provide explicit computations for the case $n = 3$, and draw a sketch with the main guidelines for the general case.

Finally, another, stronger formulation of our QDP for subgroups and homogeneous spaces can be given in terms of quantum groups of *global* type, see [CG].

1. The classical setting

In this section we introduce the notions of Poisson geometry we shall need in the following: coisotropic subgroups and Poisson quotients, also called Poisson homogeneous spaces of group type. Our aim is to stress their algebraic characterization.

1.1. Formal Poisson groups. As already explained, the setup of the paper is formal geometry. Recall that a formal variety is uniquely characterized by a tangent or a cotangent space (at its unique point), and is described by its "algebra of regular functions"—such as $F[[G]]$ below—which is a complete, topological local ring which can be realized as a $\Bbbk$-algebra of formal power series. Hereafter $\Bbbk$ is a field of zero characteristic.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\Bbbk$, and let $U(\mathfrak{g})$ be its universal enveloping algebra (with the natural Hopf algebra structure). We denote by $F[[G]]$ the algebra of functions on the formal algebraic group G associated to $\mathfrak{g}$ (which depends only on $\mathfrak{g}$ itself); this is a complete, topological Hopf algebra. One has $F[[G]] \cong U(\mathfrak{g})^*$

so that there is a natural pairing of (topological) Hopf algebras—see below—between $U(\mathfrak{g})$ and $F[[G]]$.

In general, if H, K are Hopf algebras (even topological) over a ring R, a pairing $\langle\,,\,\rangle : H \times K \longrightarrow R$ is called a *Hopf pairing* if $\langle x, y_1 \cdot y_2 \rangle = \langle \Delta(x), y_1 \otimes y_2 \rangle$, $\langle x_1 \cdot x_2, y \rangle = \langle x_1 \otimes x_2, \Delta(y) \rangle$, $\langle x, 1 \rangle = \varepsilon(x)$, $\langle 1, y \rangle = \varepsilon(y)$, $\langle S(x), y \rangle = \langle x, S(y) \rangle$ for all $x, x_1, x_2 \in H$, $y, y_1, y_2 \in K$. Moreover, a pairing is called *perfect* if it is non-degenerate.

Now assume G is a formal *Poisson* (algebraic) group. Then $\mathfrak{g}$ is a Lie bialgebra, $U(\mathfrak{g})$ is a co-Poisson Hopf algebra, $F[[G]]$ is a topological Poisson Hopf algebra, and the Hopf pairing above respects these additional co-Poisson and Poisson structures. Furthermore, the linear dual $\mathfrak{g}^*$ of $\mathfrak{g}$ is a Lie bialgebra as well, so a dual formal Poisson group G^* exists.

Notation: Hereafter, the symbol $\dot{\trianglelefteq}$ stands for "coideal", $\leq^1$ for "unital subalgebra", $\dot{\leq}$ for "subcoalgebra", $\leq_{\mathcal{P}}$ for "Poisson subalgebra", $\dot{\trianglelefteq}_{\mathcal{P}}$ for "Poisson coideal", $\leq_{\mathcal{H}}$ for "Hopf subalgebra", $\trianglelefteq_{\mathcal{H}}$ for "Hopf ideal", and the subscript ℓ stands for "left". Everything has to be meant in topological sense if necessary.

1.2. Subgroups and homogeneous G-spaces. A homogeneous left G-space M corresponds to a closed subgroup $K = K_M$, which we assume to be connected, of G such that $M \cong G/K$. Actually, in formal geometry K may be replaced by $\mathfrak{k} := Lie(K)$ as well. Then the whole geometrical setting established by the pair $\big(K, G/K\big)$ is algebraically encoded by any one of the following data:

(a) the set $\mathcal{I} = \mathcal{I}(K) \equiv \mathcal{I}(\mathfrak{k})$ of all (formal) functions vanishing on K, that is to say $\mathcal{I} = \big\{ \varphi \in F[[G]] \,\big|\, \varphi(K) = 0 \big\}$: this is a Hopf ideal of $F[[G]]$, in short $\mathcal{I} \trianglelefteq_{\mathcal{H}} F[[G]]$;

(b) the set of all left $\mathfrak{k}$-invariant functions, namely $\mathcal{C} = \mathcal{C}(K) \equiv \mathcal{C}(\mathfrak{k}) = F[[G]]^K$: this is a unital subalgebra and left coideal of $F[[G]]$, in short $\mathcal{C} \leq^1 \dot{\trianglelefteq}_\ell F[[G]]$;

(c) the set $\mathfrak{I} = \mathfrak{I}(K) \equiv \mathfrak{I}(\mathfrak{k})$ of all left-invariant differential operators on $F[[G]]$ which vanish on $F[[G]]^K$, that is $\mathfrak{I} = U(\mathfrak{g}) \cdot \mathfrak{k}$ (via standard identifications of the set of left-invariant differential operators with $U(\mathfrak{g})$): this is a left ideal and (two-sided) coideal of $U(\mathfrak{g})$, in short $\mathfrak{I}(\mathfrak{k}) = \mathfrak{I} \trianglelefteq_\ell \dot{\trianglelefteq}\, U(\mathfrak{g})$;

(d) the universal enveloping algebra of $\mathfrak{k}$, denoted $\mathfrak{C} = \mathfrak{C}(K) \equiv \mathfrak{C}(\mathfrak{k}) := U(\mathfrak{k})$: this is a Hopf subalgebra of $U(\mathfrak{g})$, i.e. $\mathfrak{C} \leq_{\mathcal{H}} U(\mathfrak{g})$.

In this way any formal subgroup K of G, or the associated homogeneous G-space G/K, is characterized—via $\mathfrak{k}$ and $\mathfrak{g}$—by any one of the following algebraic objects:

$$\text{(a)}\ \mathcal{I} \trianglelefteq_{\mathcal{H}} F[[G]], \quad \text{(b)}\ \mathcal{C} \leq^1 \dot{\trianglelefteq}_\ell F[[G]], \quad \text{(c)}\ \mathfrak{I} \trianglelefteq_\ell \dot{\trianglelefteq}\, U(\mathfrak{g}), \quad \text{(d)}\ \mathfrak{C} \leq_{\mathcal{H}} U(\mathfrak{g}). \tag{1.1}$$

Clearly (a) and (d) in (1.1) ideally focus on the subgroup K, whereas (b) and (c) focus more on the formal homogeneous G-space G/K. Nevertheless, these four algebraic data are all equivalent to each other. To express this algebraically, we need some more notation.

For any Hopf algebra H, with counit ε, and every submodule $M \subseteq H$, we set: $M^+ := M \cap Ker(\varepsilon)$ and $H^{co\,M} := \big\{ y \in H \,\big|\, \big(\Delta(y) - y \otimes 1\big) \in H \otimes M \big\}$ (the set of *M-coinvariants* of H). Letting $\mathbb{A}$ be the set of all subalgebras left coideals of H and

$\mathbb{K}$ be the set of all coideals left ideals of H, we have well-defined maps $\mathbb{A} \longrightarrow \mathbb{K}$, $A \mapsto H \cdot A^+$, and $\mathbb{K} \longrightarrow \mathbb{A}$, $K \mapsto H^{co\,K}$ (cf. [Ma], and references therein).

Then the above-mentioned equivalence stems from the following relations, which starting from any one of the four items in (1.1) allow one to reconstruct the remaining ones:

• (1) *orthogonality relations*—w.r.t. the natural pairing between $F[[G]]$ and $U(\mathfrak{g})$—namely $\mathcal{I} = \mathfrak{C}^\perp$, $\mathfrak{C} = \mathcal{I}^\perp$, linking (a) and (d), and $\mathcal{C} = \mathfrak{I}^\perp$, $\mathfrak{I} = \mathcal{C}^\perp$, linking (b) and (c);

• (2) *subgroup-space correspondence*, namely $\mathcal{I} = F[[G]] \cdot \mathcal{C}^+$, $\mathcal{C} = F[[G]]^{co\,\mathcal{I}}$, linking (a) and (b), and $\mathfrak{I} = U(\mathfrak{g}) \cdot \mathfrak{C}^+$, $\mathfrak{C} = U(\mathfrak{g})^{co\,\mathfrak{I}}$, linking (c) and (d). Moreover, the maps $\mathbb{A} \longrightarrow \mathbb{K}$ and $\mathbb{K} \longrightarrow \mathbb{A}$ considered above are inverse to each other in the formal setting.

1.3. Coisotropic subgroups and Poisson quotients. When G is a Poisson group, a distinguished class of subgroups—the *coisotropic* ones—is of special interest.

A closed formal subgroup K of G with Lie algebra $\mathfrak{k}$ is called *coisotropic* if its defining ideal $\mathcal{I}(\mathfrak{k})$ is a (topological) Poisson subalgebra of $F[[G]]$. The following are equivalent:

(C-i) K is a coisotropic formal subgroup of G;
(C-ii) $\delta(\mathfrak{k}) \subseteq \mathfrak{k} \wedge \mathfrak{g}$, that is $\mathfrak{k}$ is a Lie coideal of $\mathfrak{g}$;
(C-iii) $\mathfrak{k}^\perp$ is a Lie subalgebra of $\mathfrak{g}^*$

(see [Lu]). Clearly (C-ii) and (C-iii) characterize coisotropic subgroups in algebraic terms.

As for homogeneous spaces, recall that a formal Poisson manifold (M, ω_M) is a *Poisson homogeneous G-space* if there is a smooth homogeneous action ϕ: $G \times M \to M$ which is a Poisson map with respect to the product Poisson structure.

In addition, (M, ω_M) is said to be *of group type* (after Drinfeld [Dr2]), or simply a *Poisson quotient*, if there exists a coisotropic closed Lie subgroup K_M of G such that $G/K_M \simeq M$ and the natural projection π: $G \longrightarrow G/K_M \simeq M$ is a Poisson map.

The following is a characterization of Poisson quotients (cf. [Za]):

(PQ-i) there exists $x_0 \in M$ such that its stabilizer G_{x_0} is coisotropic in G;
(PQ-ii) there exists $x_0 \in M$ such that $\phi_{x_0} : G \longrightarrow M$, $g \mapsto \phi(g, x_0)$, is a Poisson map, that is M is a Poisson quotient;
(PQ-iii) there exists $x_0 \in M$ such that $\omega_M(x_0) = 0$.

Remark. In Poisson geometry, the usual relationship between closed subgroups of G and G-homogeneous spaces does not hold anymore. In fact, in the *same* conjugacy class one can have Poisson subgroups, coisotropic subgroups *and* non-coisotropic subgroups. We saw above that Poisson quotients correspond to Poisson homogeneous spaces in which at least one of the stabilizers is coisotropic; many such examples can be found, for instance, in [LW]. On the other hand many interesting Poisson homogeneous spaces are not of group type, as it is the case for covariant (in particular invariant) symplectic structures.

1.4. Definition. (a) If K is a formal coisotropic subgroup of G, we call complementary dual of K the formal subgroup $K^{\perp}$ of G^* whose tangent Lie algebra is $\mathfrak{k}^{\perp}$ (with G^* as in §1.1).

(b) If $M \cong G/K_M$ is a formal Poisson G-quotient, with K_M coisotropic, we call complementary dual of M the formal Poisson G^*-quotient $M^{\perp} := G^*/K_M^{\perp}$.

1.5. Remarks. (a) The fact to be highlighted in the above definition is that a subset $\mathfrak{k}$ of $\mathfrak{g}$ is a Lie coideal if and only if $\mathfrak{k}^{\perp}$ is a Lie subalgebra of $\mathfrak{g}^*$. This is why we have dual Poisson quotients. Even more, by (C-i,ii,iii) in §1.3, the complementary dual subgroup to a coisotropic subgroup is *coisotropic* too, and taking twice the complementary dual gives back the initial subgroup. Similarly, the Poisson homogeneous space which is complementary dual to a Poisson homogeneous space of group type is in turn *of group type* as well, and taking twice the complementary dual gives back the initial manifold. So Definition 1.4 makes sense, and the notion of complementary duality is self-dual, in both cases.

(b) The notion of Poisson homogeneous G-spaces of group type was first introduced by Drinfeld in [Dr2]: here the relation between such G-spaces and Lagrangian subalgebras of Drinfeld's double $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ is also explained. This is further developed in [EL].

(c) We denote by $co\mathcal{S}(G)$ the set of all formal coisotropic subgroups of G, which is as well described by the set of all Lie subalgebras, Lie coideals of $\mathfrak{g}$. This is a lattice w. r. t. set-theoretical inclusion, hence it can (and will) also be thought of as a category.

1.6. Algebraic characterization of coisotropic subgroups. Let K be a formal coisotropic subgroup of G. Taking $\mathfrak{I}$, $\mathcal{C}$, $\mathcal{I}$ and $\mathfrak{C}$ as in §1.2, coisotropy corresponds to

$$\text{(a) } \mathcal{I} \leq_{\mathcal{P}} F[[G]], \quad \text{(b) } \mathcal{C} \leq_{\mathcal{P}} F[[G]], \quad \text{(c) } \mathfrak{I} \,\dot{\trianglelefteq}_{\mathcal{P}}\, U(\mathfrak{g}), \quad \text{(d) } \mathfrak{C} \,\dot{\trianglelefteq}_{\mathcal{P}}\, U(\mathfrak{g}).$$

Thus a formal coisotropic subgroup of G is identified by any one of the algebraic objects

$$\begin{aligned} &\text{(a) } \mathcal{I} \trianglelefteq_{\mathcal{H}} \leq_{\mathcal{P}} F[[G]], \quad &&\text{(b) } \mathcal{C} \leq^1 \dot{\trianglelefteq}_{\ell} \leq_{\mathcal{P}} F[[G]], \\ &\text{(c) } \mathfrak{I} \trianglelefteq_{\ell} \dot{\trianglelefteq} \,\dot{\trianglelefteq}_{\mathcal{P}}\, U(\mathfrak{g}), \quad &&\text{(d) } \mathfrak{C} \leq_{\mathcal{H}} \dot{\trianglelefteq}_{\mathcal{P}}\, U(\mathfrak{g}). \end{aligned} \tag{1.2}$$

Note also that K being coisotropic reflects the fact that the distinguished point eK (where $e \in G$ is the identity element) in the formal Poisson G-space G/K is a zero-dimensional leaf. Then the algebra of regular functions on G/K, already realized as $F[[G]]^K$, will be also denoted by $F\big[\big[G/K\big]\big]$. Moreover, we can always choose a system of parameters for G, say $\{j_1, \ldots, j_k, j_{k+1}, \ldots, j_n\}$ such that $k = \dim(K)$, $n = \dim(G)$, $F[[G]]^K = \Bbbk[[j_{k+1}, \ldots, j_n]]$ (the topological *subalgebra* of $F[[G]]$ generated by $\{j_{k+1}, \ldots, j_n\}$) and $\mathcal{I}(K) = (j_{k+1}, \ldots, j_n)$ (the *ideal* of $F[[G]]$ generated by $\{j_{k+1}, \ldots, j_n\}$).

2. The quantum setting

This section is devoted to recall quantum groups and Drinfeld's QDP for quantum groups, to introduce our concept of quantization for coisotropic subgroups and Poisson quotients, and to explain the basic idea of our QDP for the latters.

2.1. Topological $\Bbbk[[\hbar]]$-modules and tensor structures. Let $\Bbbk[[\hbar]]$ be the topological ring of formal power series in the indeterminate $\hbar$. If X is any $\Bbbk[[\hbar]]$-module, we set $X_0 := X/\hbar X = \Bbbk \otimes_{\Bbbk[[\hbar]]} X$, the *specialization* of X at $\hbar = 0$, or *semiclassical limit* of X.

Let $\mathcal{T}_{\widehat{\otimes}}$ be the category whose objects are all topological $\Bbbk[[\hbar]]$-modules which are topologically free and whose morphisms are the $\Bbbk[[\hbar]]$-linear maps (which are automatically continuous). It is a tensor category for the tensor product $T_1 \widehat{\otimes} T_2$ defined as the separated $\hbar$-adic completion of the algebraic tensor product $T_1 \otimes_{\Bbbk[[\hbar]]} T_2$ (for all $T_1, T_2 \in \mathcal{T}_{\widehat{\otimes}}$). We denote by $\mathcal{HA}_{\widehat{\otimes}}$ the subcategory of $\mathcal{T}_{\widehat{\otimes}}$ whose objects are all the Hopf algebras in $\mathcal{T}_{\widehat{\otimes}}$ and whose morphisms are all the Hopf algebra morphisms in $\mathcal{T}_{\widehat{\otimes}}$.

Let $\mathcal{P}_{\widetilde{\otimes}}$ be the category whose objects are all topological $\Bbbk[[\hbar]]$-modules isomorphic to modules of the type $\Bbbk[[\hbar]]^E$ (with the Tikhonov product topology) for some set E, and whose morphisms are the $\Bbbk[[\hbar]]$-linear continuous maps. It is a tensor category w.r.t. the tensor product $P_1 \widetilde{\otimes} P_2$ defined as the completion of the algebraic tensor product $P_1 \otimes_{\Bbbk[[\hbar]]} P_2$ w.r.t. the weak topology: thus $P_i \cong \Bbbk[[\hbar]]^{E_i}$ $(i{=}1,2)$ yields $P_1 \widetilde{\otimes} P_2 \cong \Bbbk[[\hbar]]^{E_1 \times E_2}$ (for all $P_1, P_2 \in \mathcal{P}_{\widetilde{\otimes}}$). We call $\mathcal{HA}_{\widetilde{\otimes}}$ the subcategory of $\mathcal{P}_{\widetilde{\otimes}}$ whose objects are all the Hopf algebras in $\mathcal{P}_{\widetilde{\otimes}}$ and whose morphisms are all the Hopf algebra morphisms in $\mathcal{P}_{\widetilde{\otimes}}$.

2.2. Definition (*cf. Drinfel'd [Dr1, § 7]*). (a) We call QUEA any $H \in \mathcal{HA}_{\widehat{\otimes}}$ such that $H_0 := H/\hbar H$ is a co-Poisson Hopf algebra isomorphic to $U(\mathfrak{g})$ for some finite dimensional Lie bialgebra $(\mathfrak{g})$ (over $\Bbbk$); in this case we write $H = U_\hbar(\mathfrak{g})$, and say H is a quantization of $U(\mathfrak{g})$. We call $\mathcal{QUEA}$ the full tensor subcategory of $\mathcal{HA}_{\widehat{\otimes}}$ whose objects are QUEA, relative to all possible $\mathfrak{g}$ (see also Remark 2.3 below).

(b) We call QFSHA any $K \in \mathcal{HA}_{\widetilde{\otimes}}$ such that $K_0 := K/\hbar K$ is a topological Poisson Hopf algebra isomorphic to $F[[G]]$ for some finite dimensional formal Poisson group G (over $\Bbbk$); then we write $H = F_\hbar[[G]]$, and say K is a quantization of $F[[G]]$. We call $\mathcal{QFSHA}$ the full tensor subcategory of $\mathcal{HA}_{\widetilde{\otimes}}$ whose objects are QFSHA, relative to all possible $\mathfrak{g}$ (see also Remark 2.3 below).

2.3. Remarks. If $H \in \mathcal{HA}_{\widehat{\otimes}}$ is such that $H_0 := H/\hbar H$ as a Hopf algebra is isomorphic to $U(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$, then $H_0 = U(\mathfrak{g})$ is also a *co-Poisson* Hopf algebra w.r.t. the Poisson cobracket δ defined as follows: if $x \in H_0$ and $x' \in H$ gives $x = x' + \hbar H$, then $\delta(x) := \left(\hbar^{-1}\left(\Delta(x') - \Delta^{\mathrm{op}}(x')\right)\right) + \hbar\, H \widehat{\otimes} H$; then (by [Dr1, §3, Theorem

2]) the restriction of δ makes $\mathfrak{g}$ into a Lie bialgebra. Similarly, if $K \in \mathcal{HA}_{\tilde{\otimes}}$ is such that $K_0 := K/\hbar K$ is a topological Poisson Hopf algebra isomorphic to $F[[G]]$ for some formal group G then $K_0 = F[[G]]$ is also a topological *Poisson* Hopf algebra w.r.t. the Poisson bracket $\{\ ,\ \}$ defined as follows: if x, $y \in K_0$ and x', $y' \in K$ give $x = x' + \hbar K$, $y = y' + \hbar K$, then $\{x, y\} := \big(\hbar^{-1}(x'y' - y'x')\big) + \hbar K$; then $F[[G]]$ is (the algebra of regular functions on) a *Poisson* formal group. These natural co-Poisson and Poisson structures are the ones considered in Definition 2.2 above.

2.4. Drinfeld's functors. Let H be a (topological) Hopf algebra over $\Bbbk[[\hbar]]$. For each $n \in \mathbb{N}$, define $\Delta^n \colon H \longrightarrow H^{\otimes n}$ by $\Delta^0 := \varepsilon$, $\Delta^1 := id_H$, and $\Delta^n := \big(\Delta \otimes id_H^{\otimes(n-2)}\big) \circ \Delta^{n-1}$ if $n \geqslant 2$. For any ordered subset $E = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \cdots < i_k$, define the morphism $j_E : H^{\otimes k} \longrightarrow H^{\otimes n}$ by $j_E(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n$ with $b_i := 1$ if $i \notin \Sigma$ and $b_{i_m} := a_m$ for $1 \leqslant m \leqslant k$; then set $\Delta_E := j_E \circ \Delta^k$, $\Delta_\emptyset := \Delta^0$ and $\delta_E := \sum_{E' \subset E} (-1)^{n-|E'|} \Delta_{E'}$, $\delta_\emptyset := \varepsilon$. The inverse formula $\Delta_E = \sum_{\Psi \subseteq E} \delta_\Psi$ holds too. We shall also use the notation $\delta_0 := \delta_\emptyset$, $\delta_n := \delta_{\{1,2,\dots,n\}}$. Then we define

$$H' := \big\{ a \in H \,\big|\, \delta_n(a) \in h^n H^{\otimes n} \ \forall\, n \in \mathbb{N} \big\} \quad \big(\subseteq H \big).$$

Note that the useful formula $\delta_n = (id_H - \varepsilon)^{\otimes n} \circ \Delta^n$ holds, for all $n \in \mathbb{N}_+$. Since H splits as $H = \Bbbk[[\hbar]] \cdot 1_H \oplus J_H$, and $(id - \varepsilon)$ projects H onto $J_H := Ker(\varepsilon)$, from $\delta_n = (id_H - \varepsilon)^{\otimes n} \circ \Delta^n$ we get $\delta_n(a) = (id_H - \varepsilon)^{\otimes n}\big(\Delta^n(a)\big) \in {J_H}^{\otimes n}$ for all $a \in H$, $n \in \mathbb{N}$.

For later use, we recall that [KT, Lemma 3.2], if Φ is any finite subset of $\mathbb{N}$ then

$$\delta_\Phi(ab) = \textstyle\sum_{\Lambda \cup Y = \Phi} \delta_\Lambda(a)\, \delta_Y(b) \quad \forall\ a, b \in H; \tag{2.1}$$

$$\delta_\Phi(ab - ba) = \sum_{\substack{\Lambda \cup Y = \Phi \\ \Lambda \cap Y \neq \emptyset}} \big(\delta_\Lambda(a)\, \delta_Y(b) - \delta_Y(b)\, \delta_\Lambda(a)\big) \quad \forall a, b \in H,\ \Phi \neq \emptyset. \tag{2.2}$$

Now let $I_H := \varepsilon^{-1}\big(\hbar\, \Bbbk[[\hbar]]\big)$; set $H^\times := \sum_{n \geqslant 0} \hbar^{-n} {I_H}^n = \sum_{n \geqslant 0} \big(\hbar^{-1} I_H\big)^n = \bigcup_{n \geqslant 0} \big(\hbar^{-1} I_H\big)^n = \sum_{n \geqslant 0} \hbar^{-n} {J_H}^n$ (inside $\Bbbk((\hbar)) \otimes_{\Bbbk[[\hbar]]} H$), and define

$$H^\vee := \hbar\text{-adic completion of the } \Bbbk[[\hbar]]\text{-module } H^\times.$$

By means of this constructions, the QDP says that any QUEA provides also a QFSHA for the dual Poisson group, and any QFSHA yields also a QUEA for the dual Lie bialgebra:

2.5. Theorem (*The quantum duality principle [=QDP]; cf. Drinfel'd [Dr1, §7]; see also Etingof and Schiffman [ES, §10.2], or Gavarini [Ga1], for a proof*). *The assignments $H \mapsto H^\vee$ and $H \mapsto H'$, respectively, define tensor functors $\mathcal{QFSHA} \longrightarrow \mathcal{QUEA}$ and $\mathcal{QUEA} \longrightarrow \mathcal{QFSHA}$, which are inverse to each other. Indeed, for all $U_\hbar(\mathfrak{g}) \in \mathcal{QUEA}$*

and all $F_\hbar[[G]] \in \mathcal{QFSHA}$ *one has*

$$U_\hbar(\mathfrak{g})'/\hbar\, U_\hbar(\mathfrak{g})' = F[[G^*]], \quad F_\hbar[[G]]^\vee/\hbar\, F_\hbar[[G]]^\vee = U(\mathfrak{g}^*)$$

that is, if $U_\hbar(\mathfrak{g})$ *is a quantization of* $U(\mathfrak{g})$ *then* $U_\hbar(\mathfrak{g})'$ *is a quantization of* $F[[G^*]]$, *and if* $F_\hbar[[G]]$ *is a quantization of* $F[[G]]$ *then* $F[[G^*]]^\vee$ *is a quantization of* $U(\mathfrak{g}^*)$.

In addition, Drinfeld's functors respect Hopf duality, in the sense of the following

2.6. Proposition (*see Gavarini [Ga1, Proposition 2.2]*). *Let* $U_\hbar \in \mathcal{QUEA}$, $F_\hbar \in \mathcal{QFSHA}$ *and let* $\pi\colon U_\hbar \times F_\hbar \longrightarrow \Bbbk[[\hbar]]$ *be a perfect Hopf pairing whose specialization at* $\hbar = 0$ *is perfect as well. Then* π *induces—by restriction on l.h.s. and scalar extension on r.h.s.—a perfect Hopf pairing* ${U_\hbar}' \times {F_\hbar}^\vee \longrightarrow \Bbbk[[\hbar]]$ *whose specialization at* $\hbar = 0$ *is again perfect too.*

2.7. Quantum subgroups and quantum homogeneous spaces. From now on, let G be a formal Poisson group, $\mathfrak{g} := Lie(G)$ its tangent Lie bialgebra. We assume a quantization of G is given, in the sense that a QFSHA $F_\hbar[[G]]$ quantizing $F[[G]]$ and a QUEA $U_\hbar(\mathfrak{g})$ quantizing $U(\mathfrak{g})$ are given such that, in addition, $F_\hbar[[G]] \cong U_\hbar(\mathfrak{g})^* := Hom_{\Bbbk[[\hbar]]}\big(U_\hbar(\mathfrak{g}), \Bbbk[[\hbar]]\big)$ as topological Hopf algebras; the latter requirement is equivalent to fix a perfect Hopf algebra pairing between $F_\hbar[[G]]$ and $U_\hbar(\mathfrak{g})$ whose specialization at $\hbar = 0$ be perfect too. Note that this assumption is not restrictive: by [EK1], a QUEA $U_\hbar(\mathfrak{g})$ as required always exists, and then $F_\hbar[[G]]$ can be simply taken to be $F_\hbar[[G]] \cong U_\hbar(\mathfrak{g})^*$, by definition. Finally, as a matter of notation we denote by $\pi_{F_\hbar}\colon F_\hbar[[G]] \twoheadrightarrow F[[G]]$ and $\pi_{U_\hbar}\colon U_\hbar(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$ the specialization maps, and we set $F_\hbar := F_\hbar[[G]]$, $U_\hbar := U_\hbar(\mathfrak{g})$.

Let K be a formal subgroup of G, and $\mathfrak{k} := Lie(K)$. As quantization of K and/or of G/K, we mean a quantization of any one of the four algebraic objects $\mathcal{I}$, $\mathcal{C}$, $\mathfrak{I}$ and $\mathfrak{C}$ associated to them in §1.2, that is either of the following:

(a) a left ideal, coideal $\mathcal{I}_\hbar \trianglelefteq_\ell \dot{\trianglelefteq}\, F_\hbar[[G]]$ such that $\mathcal{I}_\hbar/\hbar\,\mathcal{I}_\hbar \cong \pi_{F_\hbar}(\mathcal{I}_\hbar) = \mathcal{I}$,

(b) a subalgebra, left coideal $\mathcal{C}_\hbar \leq^1 \dot{\trianglelefteq}_\ell\, F_\hbar[[G]]$

such that $\mathcal{C}_\hbar/\hbar\,\mathcal{C}_\hbar \cong \pi_{F_\hbar}(\mathcal{C}_\hbar) = \mathcal{C}$, (2.3)

(c) a left ideal, coideal $\mathfrak{I}_\hbar \trianglelefteq_\ell \dot{\trianglelefteq}\, U_\hbar(\mathfrak{g})$ such that $\mathfrak{I}_\hbar/\hbar\mathfrak{I}_\hbar \cong \pi_{U_\hbar}(\mathfrak{I}_\hbar) = \mathfrak{I}$,

(d) a subalgebra, left coideal $\mathfrak{C}_\hbar \leq^1 \dot{\trianglelefteq}_\ell\, U_\hbar(\mathfrak{g})$ such that $\mathfrak{C}_\hbar/\hbar\mathfrak{C}_\hbar \cong \pi_{U_\hbar}(\mathfrak{C}_\hbar) = \mathfrak{C}$.

In (2.3) the constraint $\mathcal{I}_\hbar/\hbar\,\mathcal{I}_\hbar \cong \pi_{F_\hbar}(\mathcal{I}_\hbar) = \mathcal{I}$ means the following. By construction $\mathcal{I}_\hbar \hookrightarrow F_\hbar[[G]] \xrightarrow{\pi_{F_\hbar}} F_\hbar[[G]]/\hbar\, F_\hbar[[G]] \cong F[[G]]$, and the composed map $\mathcal{I}_\hbar \longrightarrow F[[G]]$ factors through $\mathcal{I}_\hbar/\hbar\,\mathcal{I}_\hbar$; then we ask that the induced map $\mathcal{I}_\hbar/\hbar\,\mathcal{I}_\hbar \longrightarrow F[[G]]$ be a bijection onto $\pi_{F_\hbar}(\mathcal{I}_\hbar)$, and that the latter do coincide with $\mathcal{I}$; of course this bijection will also respects all Hopf operations, because $\pi_{F_\hbar}$ does. Similarly for the other conditions.

The existence of any of such objects is a separate problem, which we shall not tackle. However, the four existence problems are in fact equivalent, in that as one solves any one of them, a solution follows for the remaining ones. Indeed, much like in §1.2, one has:

• (a) $\Longleftrightarrow$ (d) and (b) $\Longleftrightarrow$ (c): if $\mathcal{I}_\hbar$ exists as in (a), then $\mathfrak{C}_\hbar := {\mathcal{I}_\hbar}^\perp$ enjoys the properties in (d); conversely, if $\mathfrak{C}_\hbar$ exists as in (d), then $\mathcal{I}_\hbar := {\mathfrak{C}_\hbar}^\perp$ enjoys the properties in (a) (hereafter orthogonality is meant w.r.t. the fixed Hopf pairing between $F_\hbar[[G]]$ and $U_\hbar(\mathfrak{g})$). The equivalence (b) $\Longleftrightarrow$ (c) follows from a like orthogonality argument.

• (a) $\Longleftrightarrow$ (b) and (c) $\Longleftrightarrow$ (d): if $\mathcal{I}_\hbar$ exists as in (a), then $\mathcal{C}_\hbar := {\mathcal{I}_\hbar}^{co\,\mathcal{I}_\hbar}$ is an object like in (b); on the other hand, if $\mathcal{C}_\hbar$ as in (b) is given, then $\mathcal{I}_\hbar := F_\hbar[[G]]{\cdot}\mathcal{C}_\hbar^+$ enjoys all properties in (a) (notation of §1.2). The equivalence (c) $\Longleftrightarrow$ (d) stems from a like argument.

From now on, *we assume from scratch that quantizations* $\mathcal{I}_\hbar$, $\mathcal{C}_\hbar$, $\mathfrak{I}_\hbar$ *and* $\mathfrak{I}_\hbar$ *as in* (2.3) *be given, and that they be linked by the like of relations* (1)–(2) *in* §1.2, *namely*

$$\begin{array}{ll} \text{(i)} \quad \mathcal{I}_\hbar = {\mathfrak{C}_\hbar}^\perp, \quad \mathfrak{C}_\hbar = {\mathcal{I}_\hbar}^\perp; & \text{(ii)} \quad \mathfrak{I}_\hbar = {\mathcal{C}_\hbar}^\perp, \quad \mathcal{C}_\hbar = {\mathfrak{I}_\hbar}^\perp; \\ \text{(iii)} \quad \mathcal{I}_\hbar = F_\hbar \cdot \mathcal{C}_\hbar^+, \quad \mathcal{C}_\hbar = {F_\hbar}^{co\,\mathcal{I}_\hbar}; & \text{(iv)} \quad \mathfrak{I}_\hbar = U_\hbar \cdot \mathfrak{C}_\hbar^+, \quad \mathfrak{C}_\hbar = {U_\hbar}^{co\,\mathfrak{I}_\hbar}. \end{array} \tag{2.4}$$

In fact, one of the objects is enough to have all the others, in such a way that the previous assumption holds. Indeed, if $co\mathcal{S} := co\mathcal{S}(G)$ let $Y_\hbar\big(co\mathcal{S}\big) := \big\{Y_\hbar(\mathfrak{k})\big\}_{\mathfrak{k}\in co\mathcal{S}}$ for all $Y\in\big\{\mathcal{I},\mathcal{C},\mathfrak{I},\mathfrak{C}\big\}$. The equivalences (a) $\Longleftrightarrow$ (d), (b) $\Longleftrightarrow$ (c), (a) $\Longleftrightarrow$ (b) and (c) $\Longleftrightarrow$ (d) seen above are given by bijective maps $\mathcal{I}_\hbar\big(co\mathcal{S}\big) \longleftrightarrow \mathfrak{C}_\hbar\big(co\mathcal{S}\big)$, $\mathcal{C}_\hbar\big(co\mathcal{S}\big) \longleftrightarrow \mathfrak{I}_\hbar\big(co\mathcal{S}\big)$, $\mathcal{I}_\hbar\big(co\mathcal{S}\big) \longleftrightarrow \mathcal{C}_\hbar\big(co\mathcal{S}\big)$ and $\mathfrak{I}_\hbar\big(co\mathcal{S}\big) \longleftrightarrow \mathfrak{C}_\hbar\big(co\mathcal{S}\big)$, respectively. Altogether these maps form a square, which happens to be *commutative*. This follows from the fact that each of these maps, or their inverse, is of type $X_\hbar \mapsto X_\hbar^\perp$, $A_\hbar \mapsto H_\hbar A_\hbar^+$ or $K_\hbar \mapsto H_\hbar^{co\,K_\hbar}$ (see §1.2): since the general relations $X_\hbar \subseteq \big(X_\hbar^\perp\big)^\perp$ and $A_\hbar \subseteq H_\hbar^{co(H_\hbar A_\hbar^+)}$ hold, and these inclusions turn to identities at $\hbar = 0$, one gets $X_\hbar = \big(X_\hbar^\perp\big)^\perp$ and $A_\hbar = H_\hbar^{co(H_\hbar A_\hbar^+)}$, which are the key steps to prove (easily) that the square of maps is commutative, as claimed.

Note also that the sets $\mathcal{I}_\hbar\big(co\mathcal{S}\big)$, $\mathfrak{C}_\hbar\big(co\mathcal{S}\big)$, $\mathcal{C}_\hbar\big(co\mathcal{S}\big)$ and $\mathfrak{I}_\hbar\big(co\mathcal{S}\big)$ are again lattices w.r.t. set theoretical inclusion, so they can (and will) be thought of as categories as well.

2.8. Remarks. (a) Let $X \in \{\mathcal{I},\mathcal{C},\mathfrak{I},\mathfrak{C}\}$ and $S_\hbar \in \big\{F_\hbar[[G]], U_\hbar(\mathfrak{g})\big\}$. Since $\pi_{S_\hbar}(X_\hbar) = X_\hbar\big/\big(X_\hbar \cap \hbar\, S_\hbar\big)$, the property $X_\hbar/\hbar\, X_\hbar \cong \pi_{S_\hbar}(X_\hbar) = X$ is equivalent to $X_\hbar \cap \hbar\, S_\hbar = \hbar\, X_\hbar$. Therefore our quantum objects can also be characterized, instead of by (2.3), by

$$\begin{array}{lll} \text{(a)}\ \mathcal{I}_\hbar \trianglelefteq_\ell \dot{\trianglelefteq}\ F_\hbar[[G]], & \mathcal{I}_\hbar \cap \hbar\, F_\hbar[[G]] = \hbar\, \mathcal{I}_\hbar, & \mathcal{I}_\hbar/\hbar\, \mathcal{I}_\hbar = \mathcal{I}, \\ \text{(b)}\ \mathcal{C}_\hbar \leq^1 \dot{\trianglelefteq}_\ell\ F_\hbar[[G]], & \mathcal{C}_\hbar \cap \hbar\, F_\hbar[[G]] = \hbar\, \mathcal{C}_\hbar, & \mathcal{C}_\hbar/\hbar\, \mathcal{C}_\hbar = \mathcal{C}, \\ \text{(c)}\ \mathfrak{I}_\hbar \trianglelefteq_\ell \dot{\trianglelefteq}\ U_\hbar(\mathfrak{g}), & \mathfrak{I}_\hbar \cap \hbar\, U_\hbar(\mathfrak{g}) = \hbar\, \mathfrak{I}_\hbar, & \mathfrak{I}_\hbar/\hbar\, \mathfrak{I}_\hbar = \mathfrak{I}, \\ \text{(d)}\ \mathfrak{C}_\hbar \leq^1 \dot{\trianglelefteq}_\ell\ U_\hbar(\mathfrak{g}), & \mathfrak{C}_\hbar \cap \hbar\, U_\hbar(\mathfrak{g}) = \hbar\, \mathfrak{C}_\hbar, & \mathfrak{C}_\hbar/\hbar\, \mathfrak{C}_\hbar = \mathfrak{C} \end{array} \tag{2.3'}$$

along with conditions (2.4). In any case, next Lemma proves that the formal subgroup of G obtained as specialization of a quantum formal subgroup is always coisotropic (much like specializing a quantum group one gets a *Poisson* group).

(b) If a quadruple $\big(\mathcal{I}_\hbar, \mathcal{C}_\hbar, \mathfrak{I}_\hbar, \mathfrak{C}_\hbar\big)$ is given which enjoys all properties in the first and the second column of (2.3)′, then one easily checks that the four specialized objects $\mathcal{I} := \mathcal{I}_\hbar\big|_{\hbar=0}$, $\mathcal{C} := \mathcal{C}_\hbar\big|_{\hbar=0}$, $\mathfrak{I} := \mathfrak{I}_\hbar\big|_{\hbar=0}$ and $\mathfrak{C} := \mathfrak{C}_\hbar\big|_{\hbar=0}$ verify relations (1) and (2) in §1.2, thus they define one single pair (*coisotropic subgroup*, *Poisson quotient*), and the quadruple $\big(\mathcal{I}_\hbar,\, \mathcal{C}_\hbar,\, \mathfrak{I}_\hbar,\, \mathfrak{C}_\hbar\big)$ then yields a quantization of the latter in the sense of §2.7.

(c) The *existence* of quantizations for a given formal coisotropic subgroup is an open question, in general. However, Etingof and Kahzdan provided a positive answer for the special subclass of those formal coisotropic subgroups K which are also *Poisson subgroups* (which infinitesimally amounts to $\mathfrak{k} := \mathit{Lie}(K)$ being a Lie subbialgebra); see [EK2, §2.2]. Several other examples of quantizations exist in literature for scattered cases of special coisotropic subgroups of interest: we shall deal with one of them in §6.

2.9. Lemma. *Let K be a formal subgroup of G, and assume a quantization $\mathcal{I}_\hbar$, $\mathcal{C}_\hbar$, $\mathfrak{I}_\hbar$ or $\mathfrak{C}_\hbar$ of $\mathcal{I}$, $\mathcal{C}$, $\mathfrak{I}$ or $\mathfrak{C}$, respectively, be given as in* §2.7. *Then K is coisotropic.*

Proof. Assume $\mathcal{I}_\hbar$ exists. Let $f, g \in \mathcal{I}$, and let $\varphi, \gamma \in \mathcal{I}_\hbar$ with $\pi_{F_\hbar}(\varphi) = f$, $\pi_{F_\hbar}(\gamma) = g$. Then by definition $\{f, g\} = \pi_{F_\hbar}\big(\hbar^{-1}[\varphi, \gamma]\big)$. But $[\varphi, \gamma] \in \mathcal{I}_\hbar \cap \hbar\, F_\hbar[G] = \hbar\, \mathcal{I}_\hbar$ by assumption, hence $\hbar^{-1}[\varphi, \gamma] \in \mathcal{I}_\hbar$, thus $\{f, g\} = \pi_{F_\hbar}\big(\hbar^{-1}[\varphi, \gamma]\big) \in \pi_{F_\hbar}(\mathcal{I}_\hbar) = \mathcal{I}$, which means that $\mathcal{I}$ is closed for the Poisson bracket. Thus (see §1.6) K is coisotropic. The proof is entirely similar when dealing with $\mathcal{C}_\hbar$, $\mathfrak{I}_\hbar$ or $\mathfrak{C}_\hbar$. $\square$

2.10. General program. Starting from the setup of §1.2, we will move along the scheme

$$
\begin{aligned}
&\text{(a)} \quad \mathcal{I}\,\big(\subseteq F[[G]]\big) \xrightarrow{(1)} \mathcal{I}_\hbar\,\big(\subseteq F_\hbar[[G]]\big) \xrightarrow{(2)} \mathcal{I}_\hbar^{\curlyvee}\,\big(\subseteq F_\hbar[[G]]^\vee\big) \\
&\qquad \xrightarrow{(3)} \mathcal{I}_0^{\curlyvee}\,\Big(\subseteq \big(F_\hbar[[G]]^\vee\big)_0 = U\big(\mathfrak{g}^*\big)\Big), \\
&\text{(b)} \quad \mathcal{C}\,\big(\subseteq F[[G]]\big) \xrightarrow{(1)} \mathcal{C}_\hbar\,\big(\subseteq F_\hbar[[G]]\big) \xrightarrow{(2)} \mathcal{C}_\hbar^{\triangledown}\,\big(\subseteq F_\hbar[[G]]^\vee\big) \\
&\qquad \xrightarrow{(3)} \mathcal{C}_0^{\triangledown}\,\Big(\subseteq \big(F_\hbar[[G]]^\vee\big)_0 = U\big(\mathfrak{g}^*\big)\Big), \\
&\text{(c)} \quad \mathfrak{I}\,\big(\subseteq U(\mathfrak{g})\big) \xrightarrow{(1)} \mathfrak{I}_\hbar\,\big(\subseteq U_\hbar(\mathfrak{g})\big) \xrightarrow{(2)} \mathfrak{I}_\hbar^{!}\,\big(\subseteq U_\hbar(\mathfrak{g})'\big) \\
&\qquad \xrightarrow{(3)} \mathfrak{I}_0^{!}\,\Big(\subseteq \big(U_\hbar(\mathfrak{g})'\big)_0 = F[[G^*]]\Big), \\
&\text{(d)} \quad \mathfrak{C}\,\big(\subseteq U(\mathfrak{g})\big) \xrightarrow{(1)} \mathfrak{C}_\hbar\,\big(\subseteq U_\hbar(\mathfrak{g})\big) \xrightarrow{(2)} \mathfrak{C}_\hbar^{\urcorner}\,\big(\subseteq U_\hbar(\mathfrak{g})'\big) \\
&\qquad \xrightarrow{(3)} \mathfrak{C}_0^{\urcorner}\,\Big(\subseteq \big(U_\hbar(\mathfrak{g})'\big)_0 = F[[G^*]]\Big).
\end{aligned}
$$

In the frame above, the arrows (1) are quantizations, as in §2.7, and the arrows (3) are specializations at $\hbar = 0$. The middle arrows (2) instead are suitable "adaptations" of Drinfeld's functors to the quantizations of K or of G/K in left hand side: roughly, one takes the suitable Drinfeld's functor on $F[[G]]$, resp. on $U(\mathfrak{g})$, and restricts it—in some sense—to the subobject $\mathcal{I}$ or $\mathcal{C}$, resp. $\mathfrak{I}$ or $\mathfrak{C}$. The points to show then are the following:

First: each one of the right-hand side objects above is one of the four algebraic objects which describe a (closed formal) subgroup of G^*: namely, the correspondence is

$$\text{(a)} \Longrightarrow \text{(c)}, \quad \text{(b)} \Longrightarrow \text{(d)}, \quad \text{(c)} \Longrightarrow \text{(a)}, \quad \text{(d)} \Longrightarrow \text{(b)}.$$

Second: all the formal subgroups of G^* associated to the four objects so obtained are *coisotropic*.

Third: the four formal subgroups of G^* in (b) do coincide.

Fourth: if we start from $K \in co\mathcal{S}(G)$, then the formal coisotropic subgroup of G^* obtained above is $K^\perp$ (cf. Definition 1.4(a)).

3. Drinfeld-like functors on quantum subgroups and Poisson quotients

In this section and next one we introduce Drinfeld-like functors for quantum coisotropic subgroups and Poisson quotients. In particular, we start with $\mathcal{I}_\hbar$, $\mathcal{C}_\hbar$, $\mathfrak{I}_\hbar$ and $\mathfrak{C}_\hbar$ as in §2.7, hence enjoying (2.3), or equivalently (2.3)$'$, and (2.4), with $F_\hbar$ and $U_\hbar$ as in §2.7. We begin moving step (2) in §2.10, with a definition whose meaning is (roughly) to "restrict" Drinfeld's functors from quantum groups to quantum subgroups or Poisson quotients:

3.1. Definition (*Drinfeld-like functors for subgroups*). Keeping notation of §2.4, we define:

(a) $\quad {\mathcal{I}_\hbar}^{\curlyvee} := \sum_{n=1}^{\infty} \hbar^{-n} \cdot I^{n-1} \cdot \mathcal{I}_\hbar = \sum_{n=1}^{\infty} \hbar^{-n} \cdot J^{n-1} \cdot \mathcal{I}_\hbar$,

(b) $\quad {\mathcal{C}_\hbar}^{\triangledown} := \mathcal{C}_\hbar + \sum_{n=1}^{\infty} \hbar^{-n} \cdot \big(\mathcal{C}_\hbar \cap I\big)^n = \Bbbk[[\hbar]] \cdot 1 + \sum_{n=1}^{\infty} \hbar^{-n} \cdot \big(\mathcal{C}_\hbar \cap J\big)^n$,

(c) $\quad {\mathfrak{I}_\hbar}^{!} := \Big\{ x \in \mathfrak{I}_\hbar \,\Big|\, \delta_n(x) \in \hbar^n \sum_{s=1}^{n} {U_\hbar}^{\widehat{\otimes}\,(s-1)} \widehat{\otimes}\, \mathfrak{I}_\hbar \,\widehat{\otimes}\, {U_\hbar}^{\widehat{\otimes}\,(n-s)},\ \forall\, n \in \mathbb{N}_+ \Big\}$,

(d) $\quad {\mathfrak{C}_\hbar}^{\Lsh} := \Big\{ x \in \mathfrak{C}_\hbar \,\Big|\, \delta_n(x) \in \hbar^n \, {U_\hbar}^{\widehat{\otimes}\,(n-1)} \widehat{\otimes}\, \mathfrak{C}_\hbar,\ \forall\, n \in \mathbb{N}_+ \Big\}$.

3.2. Remark. The following inclusion relations hold, directly by definitions:

$$\text{(i)} \quad {\mathcal{I}_\hbar}^{\curlyvee} \supseteq \mathcal{I}_\hbar, \quad \text{(ii)} \quad {\mathcal{C}_\hbar}^{\triangledown} \supseteq \mathcal{C}_\hbar, \quad \text{(iii)} \quad {\mathfrak{I}_\hbar}^{!} \subseteq \mathfrak{I}_\hbar, \quad \text{(iv)} \quad {\mathfrak{C}_\hbar}^{\Lsh} \subseteq \mathfrak{C}_\hbar.$$

Moreover, definitions and assumptions in (2.3)$'$ imply that $\mathcal{I}_\hbar = {\mathcal{I}_\hbar}^{\curlyvee} \cap F_\hbar$, $\mathcal{C}_\hbar = {\mathcal{C}_\hbar}^{\triangledown} \cap F_\hbar$, ${\mathfrak{I}_\hbar}^{!} = \mathfrak{I}_\hbar \cap {U_\hbar}'$ and ${\mathfrak{C}_\hbar}^{\Lsh} = \mathfrak{C}_\hbar \cap {U_\hbar}'$: thus we are just "restricting" Drinfeld's functors.

We can now state the QDP for formal coisotropic subgroups and Poisson quotients:

3.3. Theorem (*QDP for coisotropic subgroups and Poisson quotients*). (a) *Definition* 3.1 *provides category equivalences*

$$(\ )^{\curlyvee} \colon \mathcal{I}_\hbar\big(co\mathcal{S}(G)\big) \xrightarrow{\cong} \mathfrak{I}_\hbar\big(co\mathcal{S}(G^*)\big), \qquad (\ )^{\triangledown} \colon \mathcal{C}_\hbar\big(co\mathcal{S}(G)\big) \xrightarrow{\cong} \mathfrak{C}_\hbar\big(co\mathcal{S}(G^*)\big),$$
$$(\ )^{!} \colon \mathfrak{I}_\hbar\big(co\mathcal{S}(G)\big) \xrightarrow{\cong} \mathcal{I}_\hbar\big(co\mathcal{S}(G^*)\big), \qquad (\ )^{\Lsh} \colon \mathfrak{C}_\hbar\big(co\mathcal{S}(G)\big) \xrightarrow{\cong} \mathcal{C}_\hbar\big(co\mathcal{S}(G^*)\big),$$

along with the similar ones with G and G^* interchanged, such that $(\ )^{!} \circ (\ )^{\curlyvee} = id_{co\mathcal{S}(G)}$, $(\ )^{\curlyvee} \circ (\ )^{!} = id_{co\mathcal{S}(G^*)}$, *and* $(\ )^{\Lsh} \circ (\ )^{\triangledown} = id_{co\mathcal{S}(G)}$, $(\ )^{\triangledown} \circ (\ )^{\Lsh} = id_{co\mathcal{S}(G^*)}$, *and so on.*

(b) (*QDP*) *For any* $K \in co\mathcal{S}(G)$, *we have*

$$\mathcal{I}(\mathfrak{k})_\hbar^{\curlyvee} \bmod \hbar\, F_\hbar[[G]]^\vee = \mathfrak{I}\big(\mathfrak{k}^\perp\big), \quad \mathcal{C}(\mathfrak{k})_\hbar^{\triangledown} \bmod \hbar\, F_\hbar[[G]]^\vee = \mathfrak{C}\big(\mathfrak{k}^\perp\big),$$
$$\mathfrak{I}(\mathfrak{k})_\hbar^{!} \bmod \hbar\, U_\hbar(\mathfrak{g})' = \mathcal{I}\big(\mathfrak{k}^\perp\big), \quad \mathfrak{C}(\mathfrak{k})_\hbar^{\Lsh} \bmod \hbar\, U_\hbar(\mathfrak{g})' = \mathcal{C}\big(\mathfrak{k}^\perp\big).$$

In short, the quadruple $(\mathcal{I}(\mathfrak{k})_\hbar^{\curlyvee}, \mathcal{C}(\mathfrak{k})_\hbar^{\triangledown}, \mathfrak{I}(\mathfrak{k})_\hbar^{!}, \mathfrak{C}(\mathfrak{k})_\hbar^{\Lsh})$ *is a quantization of the quadruple* $(\mathfrak{I}(\mathfrak{k}^\perp), \mathfrak{C}(\mathfrak{k}^\perp), \mathcal{I}(\mathfrak{k}^\perp), \mathcal{C}(\mathfrak{k}^\perp))$ *w.r.t. the quantization* $(F_\hbar[[G]]^\vee, U_\hbar(\mathfrak{g})')$ *of* $(U(\mathfrak{g}^*), F[[G^*]])$.

4. First properties of Drinfeld-like functors

We shall now study the properties of the images of Drinfeld-like functors for general $\hbar$. The main result is—Proposition 4.4—that they are quantizations of some (unique) pair (*coisotropic subgroup*, *Poisson quotient*), in the sense of §2.7, for the Poisson group G^*.

4.1. Lemma. *The following relations hold* (*w.r.t. the perfect Hopf pairing between* ${U_\hbar}'$ *and* ${F_\hbar}^\vee$ *given by Proposition* 2.6 *for the orthogonality relations* (i)–(ii)):

$$\text{(i) } {\mathcal{I}_\hbar}^{\curlyvee} = \big({\mathfrak{C}_\hbar}^{\Lsh}\big)^\perp, \quad {\mathfrak{C}_\hbar}^{\Lsh} = \big({\mathcal{I}_\hbar}^{\curlyvee}\big)^\perp, \quad \text{(ii) } {\mathfrak{I}_\hbar}^{!} = \big({\mathcal{C}_\hbar}^{\triangledown}\big)^\perp, \quad {\mathcal{C}_\hbar}^{\triangledown} = \big({\mathfrak{I}_\hbar}^{!}\big)^\perp,$$
$$\text{(iii) } {\mathcal{I}_\hbar}^{\curlyvee} = {F_\hbar}^\vee \cdot \big({\mathcal{C}_\hbar}^{\triangledown}\big)^+, \ \ {\mathcal{C}_\hbar}^{\triangledown} = \big({F_\hbar}^\vee\big)^{co\,{\mathcal{I}_\hbar}^{\curlyvee}}, \quad \text{(iv) } {\mathfrak{I}_\hbar}^{!} = {U_\hbar}' \cdot \big({\mathfrak{C}_\hbar}^{\Lsh}\big)^+, \ \ {\mathfrak{C}_\hbar}^{\Lsh} = \big({U_\hbar}'\big)^{co\,{\mathfrak{I}_\hbar}^{!}}.$$

Proof. Let $I = I_{F_\hbar}$ be the ideal of $F_\hbar$ considered in §2.4, and take $y_1, \ldots, y_{n-1} \in I$; then $\langle y_i, 1\rangle = \varepsilon(y_i) \in \hbar \cdot \Bbbk[[\hbar]]$, for all $i = 1, \ldots, n-1$. Given $y_n \in \mathcal{I}_\hbar$ and $\gamma \in {\mathfrak{C}_\hbar}^{\Lsh}$, consider

$$\left\langle \prod_{i=1}^{n} y_i, \gamma \right\rangle = \left\langle \bigotimes_{i=1}^{n} y_i, \Delta^n(\gamma) \right\rangle = \left\langle \bigotimes_{i=1}^{n} y_i, \sum_{\Psi \subseteq \{1,\ldots,n\}} \delta_\Psi(\gamma) \right\rangle = \sum_{\Psi \subseteq \{1,\ldots,n\}} \left\langle \bigotimes_{i=1}^{n} c_i, \delta_\Psi(\gamma) \right\rangle.$$

Now consider any summand in the last term in the formula above. Let $|\Psi| = t$ ($t \leqslant n$): then $\left\langle \otimes_{i=1}^{n} y_i, \delta_\Psi(\gamma) \right\rangle = \left\langle \otimes_{i\in\Psi} y_i, \delta_t(\gamma) \right\rangle \cdot \prod_{j\notin\Psi} \langle y_j, 1\rangle$, by definition of δ_Ψ.

Thanks to the previous analysis, we have $\prod_{j \notin \Psi} \langle y_j, 1 \rangle \in \hbar^{n-t} \Bbbk[[\hbar]]$, hence

$$\left\langle \bigotimes_{i \in \Psi} y_i, \, \delta_t(\gamma) \right\rangle \in \left\langle \bigotimes_{i \in \Psi} y_i, \, \hbar^t \sum_{s=1}^n {U_\hbar}^{\widehat{\otimes}(n-1)} \widehat{\otimes} \mathfrak{C}_\hbar \right\rangle \subseteq \hbar^{t+1} \Bbbk[[\hbar]]$$

because $\gamma \in {\mathfrak{C}_\hbar}^{\curlyvee}$; therefore $\left\langle \prod_{i=1}^n y_i, \, \gamma \right\rangle \in \hbar \, \Bbbk[[\hbar]]$. And even more, the rightmost tensor factor in each summand $\delta_\Psi(\gamma)$ always belongs to $\mathfrak{C}_\hbar$ (as also $1 \in \mathfrak{C}_\hbar$), whereas $y_n \in \mathcal{I}_\hbar = {\mathfrak{C}_\hbar}^\perp$: therefore $\left\langle \prod_{i=1}^n y_i, \, \gamma \right\rangle = \left\langle \bigotimes_{i=1}^n y_i, \sum_{\Psi \subseteq \{1,\dots,n\}} \delta_\Psi(\gamma) \right\rangle = 0$. This means that

$$\mathcal{I}_\hbar{}^\curlyvee \subseteq \left({\mathfrak{C}_\hbar}^{\curlyvee}\right)^\perp, \quad {\mathfrak{C}_\hbar}^{\curlyvee} \subseteq \left(\mathcal{I}_\hbar{}^\curlyvee\right)^\perp. \tag{4.1}$$

Now take $\kappa \in \left(\mathcal{I}_\hbar{}^\curlyvee\right)^\perp \subseteq \left(F_\hbar{}^\vee\right)^* = {U_\hbar}'$ (using Proposition 2.6 for the last equality). Since $\kappa \in {U_\hbar}'$, we have $\delta_n(\kappa) \in \hbar^n \, {U_\hbar}^{\widehat{\otimes} n}$ for all $n \in \mathbb{N}$, and moreover from $\kappa \in \left(\mathcal{I}_\hbar{}^\curlyvee\right)^\perp$ it follows that $\kappa_+ := \hbar^{-n} \, \delta_n(\kappa)$ enjoys $\left\langle I^{\widetilde{\otimes}(n-1)} \widetilde{\otimes} \mathcal{I}_\hbar, \, \kappa_+ \right\rangle = 0$, so that

$$\kappa_+ \in \left(I^{\widetilde{\otimes}(n-1)} \widetilde{\otimes} \mathcal{I}_\hbar \right)^\perp = \sum_{r+s=n-2} {U_\hbar}^{\widehat{\otimes} r} \widehat{\otimes} I^\perp \widehat{\otimes} {U_\hbar}^{\widehat{\otimes} s} \widehat{\otimes} U_\hbar \, + \, {U_\hbar}^{\widehat{\otimes}(n-1)} \widehat{\otimes} {\mathcal{I}_\hbar}^\perp.$$

In addition, $\delta_n(\kappa) \in J^{\widehat{\otimes} n}$, where $J := J_{U_\hbar} = Ker\left(\varepsilon \colon U_\hbar \longrightarrow \Bbbk[[\hbar]]\right)$, hence $\delta_n(\kappa) \in \hbar^n \, {U_\hbar}^{\widehat{\otimes} n} \cap J^{\widehat{\otimes} n} = \hbar^n \, J^{\widehat{\otimes} n}$; this together with the above formula yields

$$\begin{aligned}
\kappa_+ &\in \left(I^{\widetilde{\otimes}(n-1)} \widetilde{\otimes} \mathcal{I}_\hbar \right)^\perp \cap J^{\widehat{\otimes} n} \\
&= \left(\sum_{r+s=n-2} {U_\hbar}^{\widehat{\otimes} r} \otimes I^\perp \widehat{\otimes} {U_\hbar}^{\widehat{\otimes} s} \widehat{\otimes} U_\hbar \right) \cap J^{\widehat{\otimes} n} + \left({U_\hbar}^{\widehat{\otimes}(n-1)} \widehat{\otimes} {\mathcal{I}_\hbar}^\perp \right) \cap J^{\widehat{\otimes} n} \\
&= \sum_{r+s=n-2} J^{\widehat{\otimes} r} \widehat{\otimes} \left(I^\perp \cap J_U \right) \widehat{\otimes} J^{\widehat{\otimes} s} \widehat{\otimes} J + J^{\widehat{\otimes}(n-1)} \widehat{\otimes} \left({\mathcal{I}_\hbar}^\perp \cap J \right) \\
&= J^{\widehat{\otimes}(n-1)} \widehat{\otimes} \left({\mathcal{I}_\hbar}^\perp \cap J \right) = J^{\widehat{\otimes}(n-1)} \widehat{\otimes} \left(\mathfrak{C}_\hbar \cap J \right) \subseteq {U_\hbar}^{\widehat{\otimes}(n-1)} \widehat{\otimes} \mathfrak{C}_\hbar,
\end{aligned}$$

where in the third equality we used the fact that $I^\perp = 0$; the last equality then follows from (2.4)(i). Thus $\kappa_+ \in {U_\hbar}^{\widehat{\otimes}(n-1)} \widehat{\otimes} \mathfrak{C}_\hbar$, hence $\delta_n(\kappa) \in \hbar^n \, {U_\hbar}^{\widehat{\otimes}(n-1)} \widehat{\otimes} \mathfrak{C}_\hbar$ for all $n \in \mathbb{N}$: so $\kappa \in {\mathfrak{C}_\hbar}^{\curlyvee}$. We conclude that $\left(\mathcal{I}_\hbar{}^\curlyvee\right)^\perp \subseteq {\mathfrak{C}_\hbar}^{\curlyvee}$, which together with (4.1) gives ${\mathfrak{C}_\hbar}^{\curlyvee} = \left(\mathcal{I}_\hbar{}^\curlyvee\right)^\perp$.

By Proposition 2.6 the specialization at $\hbar = 0$ of the pairing between ${U_\hbar}'$ and $F_\hbar{}^\vee$ is perfect too. From this we can easily argue that $\mathcal{I}_\hbar{}^\curlyvee \equiv \left(\left(\mathcal{I}_\hbar{}^\curlyvee\right)^\perp\right)^\perp \bmod \hbar \, F_\hbar{}^\vee$, whence $\mathcal{I}_\hbar{}^\curlyvee = \left(\left(\mathcal{I}_\hbar{}^\curlyvee\right)^\perp\right)^\perp$ follows at once by $\hbar$-adic completeness. But then starting from

${\mathfrak{C}_\hbar}^{\urcorner} = \big({\mathcal{I}_\hbar}^{\curlyvee}\big)^{\perp}$, hence $\big({\mathfrak{C}_\hbar}^{\urcorner}\big)^{\perp} = \Big(\big({\mathcal{I}_\hbar}^{\curlyvee}\big)^{\perp}\Big)^{\perp}$, we finally get $\big({\mathfrak{C}_\hbar}^{\urcorner}\big)^{\perp} = {\mathcal{I}_\hbar}^{\curlyvee}$, thus (i) is proved.

The proof of (ii) is similar. First of all, by (2.4)(ii) and definitions it is clear that

$$ {\mathfrak{I}_\hbar}^{!} \subseteq \big({\mathcal{C}_\hbar}^{\triangledown}\big)^{\perp}, \quad {\mathcal{C}_\hbar}^{\triangledown} \subseteq \big({\mathfrak{I}_\hbar}^{!}\big)^{\perp}. \tag{4.2} $$

Now notice that ${\mathcal{C}_\hbar}^{\triangledown} \supseteq \mathcal{C}_\hbar$, so $\big({\mathcal{C}_\hbar}^{\triangledown}\big)^{\perp} \subseteq {\mathcal{C}_\hbar}^{\perp} = \mathfrak{I}_\hbar$, due to (2.4)(ii); thus $\big({\mathcal{C}_\hbar}^{\triangledown}\big)^{\perp} \subseteq \mathfrak{I}_\hbar$. Second, pick $\eta \in \big({\mathcal{C}_\hbar}^{\triangledown}\big)^{\perp}$ $\big(\subseteq {U_\hbar}'\big)$. Then $\delta_n(\eta) \in \hbar^n {U_\hbar}^{\widehat{\otimes}\, n}$ for all $n \in \mathbb{N}_+$, and from $\eta \in \big({\mathcal{C}_\hbar}^{\triangledown}\big)^{\perp}$ we get that $\eta_+ := \hbar^{-n}\,\delta_n(\eta)$ enjoys $\Big\langle \big(\mathcal{C}_\hbar \cap I\big)^{\widetilde{\otimes}\, n},\, \eta_+ \Big\rangle = 0$, so that

$$ \eta_+ \in \Big(\big(\mathcal{C}_\hbar \cap I\big)^{\widetilde{\otimes}\, n}\Big)^{\perp} = \sum_{r+s=n-1} {U_\hbar}^{\widehat{\otimes}\, r} \,\widehat{\otimes}\, \big(\mathcal{C}_\hbar \cap I\big)^{\perp} \,\widehat{\otimes}\, {U_\hbar}^{\widehat{\otimes}\, s}. $$

Moreover $\delta_n(\eta) \in J^{\widehat{\otimes}\, n}$, hence $\delta_n(\eta) \in \hbar^n\, {U_\hbar}^{\widehat{\otimes}\, n} \cap J^{\widehat{\otimes}\, n} = \hbar^n J^{\widehat{\otimes}\, n}$, so $\eta_+ \in J^{\widehat{\otimes}\, n}$ and

$$ \eta_+ \in \Big(\big(\mathcal{C}_\hbar \cap I\big)^{\widehat{\otimes}\, n}\Big)^{\perp} \cap J^{\widehat{\otimes}\, n} = \Big(\textstyle\sum_{r+s=n-1} {U_\hbar}^{\widehat{\otimes}\, r} \,\widehat{\otimes}\, \big(\mathcal{C}_\hbar \cap I\big)^{\perp} \,\widehat{\otimes}\, {U_\hbar}^{\widehat{\otimes}\, s}\Big) \cap J^{\widehat{\otimes}\, n} $$
$$ = \textstyle\sum_{r+s=n-1} J^{\widehat{\otimes}\, r} \,\widehat{\otimes}\, \Big(\big(\mathcal{C}_\hbar \cap I\big)^{\perp} \cap J\Big) \,\widehat{\otimes}\, J^{\widehat{\otimes}\, s}. $$

Now $\big(\mathcal{C}_\hbar \cap I\big)^{\perp} \cap J = {\mathcal{C}_\hbar}^{\perp} \cap J = \mathfrak{I}_\hbar \cap J \subseteq \mathfrak{I}_\hbar$, thanks to (2.4)(ii). The upshot is

$$ \eta_+ \in \textstyle\sum_{r+s=n-1} J^{\widehat{\otimes}\, r} \,\widehat{\otimes}\, \big(\mathfrak{I}_\hbar \cap J_U\big) \,\widehat{\otimes}\, J^{\widehat{\otimes}\, s} \subseteq \textstyle\sum_{r+s=n-1} {U_\hbar}^{\widehat{\otimes}\, r} \,\widehat{\otimes}\, \mathfrak{I}_\hbar \,\widehat{\otimes}\, {U_\hbar}^{\widehat{\otimes}\, s} $$

whence we get $\delta_n(\eta) \in \hbar^n \sum_{r+s=n-1} {U_\hbar}^{\widehat{\otimes}\, r} \,\widehat{\otimes}\, \mathfrak{I}_\hbar \,\widehat{\otimes}\, {U_\hbar}^{\widehat{\otimes}\, s}$ for all $n \in \mathbb{N}_+$. Since in addition $\eta \in \mathfrak{I}_\hbar$, for we proved that $\big({\mathcal{C}_\hbar}^{\triangledown}\big)^{\perp} \subseteq \mathfrak{I}_\hbar$, we argue that $\eta \in {\mathfrak{I}_\hbar}^{!}$. The final outcome is $\big({\mathcal{C}_\hbar}^{\triangledown}\big)^{\perp} \subseteq {\mathfrak{I}_\hbar}^{!}$, which together with (4.2) implies ${\mathfrak{I}_\hbar}^{!} = \big({\mathcal{C}_\hbar}^{\triangledown}\big)^{\perp}$.

With like arguments as for part (i) one proves that $\Big(\big({\mathcal{C}_\hbar}^{\triangledown}\big)^{\perp}\Big)^{\perp} = {\mathcal{C}_\hbar}^{\triangledown}$ and then argue that $\big({\mathfrak{I}_\hbar}^{!}\big)^{\perp} = {\mathcal{C}_\hbar}^{\triangledown}$; this ends the proof of claim (ii) too. Finally, (iii) and (iv) are straightforward consequence of relations (iii) and (iv) in (2.4) and of definitions. □

4.2. Lemma.

(a) ${\mathcal{I}_\hbar}^{\curlyvee} \trianglelefteq_\ell {F_\hbar}^{\vee}$, (b) ${\mathcal{C}_\hbar}^{\triangledown} \leq^1 {F_\hbar}^{\vee}$, (c) ${\mathfrak{I}_\hbar}^{!} \trianglelefteq_\ell {U_\hbar}'$, (d) ${\mathfrak{C}_\hbar}^{\urcorner} \leq^1 {U_\hbar}'$,

(e) ${\mathcal{I}_\hbar}^{\curlyvee} \,\dot{\trianglelefteq}\, {F_\hbar}^{\vee}$, (f) ${\mathcal{C}_\hbar}^{\triangledown} \,\dot{\trianglelefteq}_\ell\, {F_\hbar}^{\vee}$, (g) ${\mathfrak{I}_\hbar}^{!} \,\dot{\trianglelefteq}\, {U_\hbar}'$, (h) ${\mathfrak{C}_\hbar}^{\urcorner} \,\dot{\trianglelefteq}_\ell\, {U_\hbar}'$.

Proof. The statements on the first line are proved directly, and imply those on the second line via the orthogonality relations of Lemma 4.1.

Claim (a) is straightforward, and (b) follows directly from definitions. To prove (c), let $a \in {U_\hbar}'$ and $b \in {\mathfrak{I}_\hbar}^!$: by definition of ${\mathfrak{I}_\hbar}^!$, from $\mathfrak{I}_\hbar \trianglelefteq_\ell U_\hbar$ and from (2.1) we get $\delta_n(ab) \in \hbar^n \sum_{s=1}^n {U_\hbar}^{\widehat{\otimes}\,(s-1)} \widehat{\otimes}\, \mathfrak{I}_\hbar \,\widehat{\otimes}\, {U_\hbar}^{\widehat{\otimes}\,(n-s)}$, so $ab \in {\mathfrak{I}_\hbar}^!$; thus ${\mathfrak{I}_\hbar}^! \trianglelefteq_\ell {U_\hbar}'$. Recall that ${U_\hbar}'$ is commutative modulo $\hbar$, and $\hbar\, {U_\hbar}' \in {\mathfrak{I}_\hbar}^!$: then ${\mathfrak{I}_\hbar}^! \trianglelefteq_\ell {U_\hbar}'$ implies ${\mathfrak{I}_\hbar}^! \trianglelefteq {U_\hbar}'$ (a two-sided ideal), thus proving (c). Lastly, to prove (d), remark that $1 \in \mathfrak{C}_\hbar$ and $\delta_n(1) = 0$ for all $n \in \mathbb{N}$, so $1 \in {\mathfrak{C}_\hbar}^{\Lsh}$. Let $x, y \in {\mathfrak{C}_\hbar}^{\Lsh}$ and $n \in \mathbb{N}$; by (2.1) we have $\delta_n(xy) = \sum_{\Lambda \cup Y = \{1,\dots,n\}} \delta_\Lambda(x)\, \delta_Y(y)$. Each of the factors $\delta_\Lambda(x)$ belongs to a module $\hbar^{|\Lambda|}\, {U_\hbar}^{\widehat{\otimes}\,(|\Lambda|-1)} \widehat{\otimes}\, X$ where the last tensor factor is either $X = \mathfrak{C}_\hbar$ (if $n \in \Lambda$) or $X = \{1\} \subset \mathfrak{C}_\hbar$ (if $n \notin \Lambda$), and similarly for $\delta_Y(y)$; but $\Lambda \cup Y = \{1, \dots, n\}$ implies $|\Lambda| + |Y| \geqslant n$, and summing up $\delta_n(xy) \in \hbar^n {U_\hbar}^{\widehat{\otimes}\,(n-1)} \widehat{\otimes}\, \mathfrak{C}_\hbar$, whence $xy \in {\mathfrak{C}_\hbar}^{\Lsh}$. Thus ${\mathfrak{C}_\hbar}^{\Lsh} \leq^1 {U_\hbar}'$. $\square$

Remark. In the previous proof one might also prove the required properties for only one of the objects involved, say ${\mathfrak{I}_\hbar}^!$ for instance: then the properties of all others objects will follow from relations (i)–(iv) in Lemma 4.1.

4.3. Lemma.

$$\text{(a)}\ \ {\mathcal{I}_\hbar}^{\curlyvee} \bigcap \hbar\, {F_\hbar}^{\vee} = \hbar\, {\mathcal{I}_\hbar}^{\curlyvee}, \qquad \text{(b)}\ \ {\mathcal{C}_\hbar}^{\triangledown} \bigcap \hbar\, {F_\hbar}^{\vee} = \hbar\, {\mathcal{C}_\hbar}^{\triangledown},$$
$$\text{(c)}\ \ {\mathfrak{I}_\hbar}^! \bigcap \hbar\, {U_\hbar}' = \hbar\, {\mathfrak{I}_\hbar}^!, \qquad \text{(d)}\ \ {\mathfrak{C}_\hbar}^{\Lsh} \bigcap \hbar\, {U_\hbar}' = \hbar\, {\mathfrak{C}_\hbar}^{\Lsh}.$$

Proof. We start proving claim (c). Let $\eta \in {\mathfrak{I}_\hbar}^! \cap \hbar\, {U_\hbar}' = \hbar\, {\mathfrak{I}_\hbar}^!$. Then

$$\delta_n(\eta) \in \hbar^n \left(\left(\sum_{s=1}^n {U_\hbar}^{\widehat{\otimes}\,(s-1)} \,\widehat{\otimes}\, \mathfrak{I}_\hbar \,\widehat{\otimes}\, {U_\hbar}^{\widehat{\otimes}\,(n-s)} \right) \bigcap \hbar\, {U_\hbar}^{\widehat{\otimes}\, n} \right) \tag{4.3}$$

for all $n \in \mathbb{N}_+$. Now, for $n \in \mathbb{N}_+$ we have $\left(\sum_{s=1}^n {U_\hbar}^{\widehat{\otimes}\,(s-1)} \,\widehat{\otimes}\, \mathfrak{I}_\hbar \,\widehat{\otimes}\, {U_\hbar}^{\widehat{\otimes}\,(n-s)} \right) \bigcap \hbar\, {U_\hbar}^{\widehat{\otimes}\, n}$ $= \sum_{s=1}^n {U_\hbar}^{\widehat{\otimes}\,(s-1)} \,\widehat{\otimes} \left(\mathfrak{I}_\hbar \bigcap \hbar\, U_\hbar \right) \widehat{\otimes}\, {U_\hbar}^{\widehat{\otimes}\,(n-s)}$, and since $\mathfrak{I}_\hbar \bigcap \hbar\, U_\hbar = \hbar\, \mathfrak{I}_\hbar$ by (2.3)$'$, from (4.3) we conclude that $\delta_n(\eta) \in \hbar^{n+1} \sum_{s=1}^n {U_\hbar}^{\widehat{\otimes}\,(s-1)} \,\widehat{\otimes}\, \mathfrak{I}_\hbar \,\widehat{\otimes}\, {U_\hbar}^{\widehat{\otimes}\,(n-s)}$ for all $n \in \mathbb{N}_+$, which in turn means $\eta \in \hbar\, {\mathfrak{I}_\hbar}^!$.

The converse inclusion ${\mathfrak{I}_\hbar}^! \cap \hbar\, {U_\hbar}' \supseteq \hbar\, {\mathfrak{I}_\hbar}^!$ is trivially true. The same arguments prove (d) as well.

As for (a) and (b), we can give a rather concrete description of the objects involved, starting from ${F_\hbar}^{\vee}$. Let $I := I_{F_\hbar}$ as in §2.4, $J := \mathit{Ker}\big(\varepsilon \colon F_\hbar \longrightarrow \Bbbk[[\hbar]]\big)$, and $J^{\vee} := \hbar^{-1} J \subset {F_\hbar}^{\vee}$. Then $J \bmod \hbar\, F_\hbar = J_G := \mathit{Ker}\big(\varepsilon \colon F[[G]] \to \Bbbk\big)$, and $J_G / {J_G}^2 = \mathfrak{g}^*$. Let $\{y_1, \dots, y_n\}$, with $n := \dim(G)$, be a $\Bbbk$-basis of $J_G / {J_G}^2$, and pull it back to a subset $\{j_1, \dots, j_n\}$ of J. Then $\left\{ \hbar^{-|\underline{e}|} j^{\,\underline{e}} \bmod \hbar\, {F_\hbar}^{\vee} \,\middle|\, \underline{e} \in \mathbb{N}^{\,n} \right\}$ (with $j^{\,\underline{e}} := \prod_{s=1}^n j_s^{\underline{e}(i)}$, and similarly hereafter) is a $\Bbbk$-basis of ${F_0}^{\vee}$ and, setting $j_s^{\vee} := \hbar^{-1} j_s$ for all s, the set $\left\{ j_1^{\vee}, \dots, j_n^{\vee} \right\}$ is a $\Bbbk$-basis of $\mathfrak{t} := J^{\vee} \bmod \hbar\, {F_\hbar}^{\vee}$. Moreover, since $j_\mu\, j_\nu - j_\nu\, j_\mu \in \hbar\, J$ (for $\mu, \nu \in \{1, \dots, n\}$) we have $j_\mu\, j_\nu - j_\nu\, j_\mu = \hbar \sum_{s=1}^n c_s\, j_s + \hbar^2 \gamma_1 + \hbar\, \gamma_2$ for some $c_s \in \Bbbk[[\hbar]]$, $\gamma_1 \in J$ and $\gamma_2 \in J^2$, whence $\left[j_\mu^{\vee}, j_\nu^{\vee} \right] := j_\mu^{\vee}\, j_\nu^{\vee} - j_\nu^{\vee}\, j_\mu^{\vee} \equiv \sum_{s=1}^n c_s\, j_s^{\vee} \bmod \hbar\, {F_\hbar}^{\vee}$, thus $\mathfrak{t} := J^{\vee} \bmod \hbar\, {F_\hbar}^{\vee}$ is a Lie subalgebra of ${F_0}^{\vee}$: indeed, ${F_0}^{\vee} = U(\mathfrak{t})$ as Hopf algebras.

Now for the second step. The specialization map $\pi^\vee \colon {F_\hbar}^\vee \twoheadrightarrow {F_0}^\vee = U(\mathfrak{t})$ restricts to $\eta \colon J^\vee \twoheadrightarrow \mathfrak{t} := J^\vee \bmod \hbar\, {F_\hbar}^\vee = J^\vee / J^\vee \cap \big(\hbar\, {F_\hbar}^\vee\big) = J^\vee / \big(J + J^\vee J_\hbar\big)$, because $J^\vee \cap \big(\hbar\, {F_\hbar}^\vee\big) = J^\vee \cap \hbar^{-1} I_{F_\hbar}^{\,2} = J_\hbar + J^\vee J_\hbar$. Moreover, multiplication by $\hbar^{-1}$ yields a $\Bbbk[[\hbar]]$-module isomorphism $\mu \colon J \overset{\cong}{\hookrightarrow\!\!\!\!\rightarrow} J^\vee$. Let $\rho \colon J_G \twoheadrightarrow J_G / J_G^{\,2} = \mathfrak{g}^*$ be the natural projection map, and $\nu \colon \mathfrak{g}^* \hookrightarrow J_G$ a section of ρ. The specialization map $\pi \colon F_\hbar \twoheadrightarrow F_0$ restricts to $\pi' \colon J \twoheadrightarrow J / (J \cap \hbar\, F_\hbar) = J_\hbar / \hbar\, J_\hbar = J_G$: we fix a section $\gamma \colon J_G \hookrightarrow J_\hbar$ of π'. Then the composition map $\sigma := \eta \circ \mu \circ \gamma \circ \nu \colon \mathfrak{g}^* \longrightarrow \mathfrak{t}$ is a well-defined Lie bialgebra isomorphism, independent of the choice of ν and γ. In fact, one has (see [Ga1]) $F_\hbar[[G]] \cong \big(\Bbbk[[j_1, \dots, j_n]]\big)[[\hbar]]$ and $U_\hbar(\mathfrak{g}) \cong \big(\Bbbk\big[j_1^\vee, \dots, j_n^\vee\big]\big)[[\hbar]]$ as topological $\Bbbk[[\hbar]]$-modules.

For our purposes we need a special choice of the $\Bbbk$-basis $\{y_1, \dots, y_n\}$ of $\mathfrak{g}^* = J_G / J_G^{\,2}$. Namely, letting $k := \dim(K)$, we fix a system of parameters $\{j_1, \dots, j_k, j_{k+1}, \dots, j_n\}$ for $F[[G]]$ like in the end of §1.6: then in particular $\big(\{j_{k+1}, \dots, j_n\} \bmod {J_G}^2\big) \bmod \mathfrak{k}^*$ is a $\Bbbk$-basis of $\mathfrak{g}^*/\mathfrak{k}^* = \mathfrak{k}^\perp$, the cotangent space of G/K at the point eK.

By construction $\big(\mathcal{I} + J_G^{\,2}\big) \cap \mathit{Span}\big(\{j_1, \dots, j_k\}\big) = \{0\}$ and $\rho(\mathcal{I}) = \big(\mathcal{I} + J_G^{\,2}\big) \bmod J_G^{\,2} = \mathit{Span}\big(\{y_{k+1}, \dots, y_n\}\big) = \mathfrak{k}^\perp$. Thus we choose *this* set $\big\{y_1, \dots, y_k, y_{k+1}, \dots, y_n\big\}$ as the basis of $J_G / J_G^{\,2} = \mathfrak{g}^*$ to start with. Then $\mathcal{I}_\hbar$ identifies with the left ideal of $F_\hbar[[G]] = \big(\Bbbk[[j_1, \dots, j_n]]\big)[[\hbar]]$ generated by $\{j_{k+1}, \dots, j_n\}$, which is the set of all formal power series in $\{j_1, \dots, j_n, \hbar\}$ such that in each monomial with non-zero coefficient at least one out of $j_{k+1}, \dots, j_n$ does occur with non-zero exponent. Similarly, ${\mathcal{I}_\hbar}^{\curlyvee}$ identifies with the left ideal of $U_\hbar(\mathfrak{g}) = \big(\Bbbk\big[j_1^\vee, \dots, j_n^\vee\big]\big)[[\hbar]]$ generated by $\big\{j_{k+1}^\vee, \dots, j_n^\vee\big\}$, which is the set of all formal power series in $\hbar$ with coefficients in $\Bbbk\big[j_1^\vee, \dots, j_n^\vee\big]$ such that in each monomial in the $j_r^\vee$'s with non-zero coefficient at least one out of $j_{k+1}^\vee, \dots, j_n^\vee$ occurs with non-zero exponent. But then it is clear—thanks to (2.3)′—that ${\mathcal{I}_\hbar}^{\curlyvee} \cap \hbar\, F_\hbar[G]^\vee \subseteq \hbar\, {\mathcal{I}_\hbar}^{\curlyvee}$. The converse inclusion ${\mathcal{I}_\hbar}^{\curlyvee} \cap \hbar\, F_\hbar[G]^\vee \supseteq \hbar\, {\mathcal{I}_\hbar}^{\curlyvee}$ is obvious. Similarly one proves (b). $\square$

Altogether, Lemmas 4.1–4.3 yield the main result of this section, namely

4.4. Proposition. *${\mathcal{I}_\hbar}^{\curlyvee}$, ${\mathfrak{C}_\hbar}^{\urcorner}$, ${\mathcal{C}_\hbar}^{\triangledown}$ and ${\mathfrak{I}_\hbar}^{!}$ are quantizations of a pair* (*coisotropic subgroup*, *Poisson quotient*), *in the sense of* §2.7, *for the dual Poisson group G^*.*

Next result instead shows that the construction by Drinfeld-like functors is involutive:

4.5. Proposition. *The following identities hold*:

$$\big({\mathcal{I}_\hbar}^{\curlyvee}\big)^{!} = \mathcal{I}_\hbar, \quad \big({\mathcal{C}_\hbar}^{\triangledown}\big)^{\urcorner} = \mathcal{C}_\hbar, \quad \big({\mathfrak{I}_\hbar}^{!}\big)^{\curlyvee} = \mathfrak{I}_\hbar, \quad \big({\mathfrak{C}_\hbar}^{\urcorner}\big)^{\triangledown} = \mathfrak{C}_\hbar.$$

Proof. From the very definitions we get

$$\delta_n\big(\mathcal{I}_\hbar\big) \;\subseteq\; \sum\nolimits_{s=1}^{n} J_{F_\hbar}^{\;\widetilde{\otimes}\,(s-1)} \,\widetilde{\otimes}\, \mathcal{I}_\hbar \,\widetilde{\otimes}\, J_{F_\hbar}^{\;\widetilde{\otimes}\,(n-s)} \;\subseteq\; \sum\nolimits_{s=1}^{n} \Big(\hbar^{s-1} \big({F_\hbar}^\vee\big)^{\widehat{\otimes}\,(s-1)}\Big)$$

$$\widehat{\otimes}\left(\hbar\,\mathcal{I}_\hbar^{\curlyvee}\right)\widehat{\otimes}\left(\hbar^{n-s}\left(F_\hbar{}^{\vee}\right)^{\widehat{\otimes}\,(n-s)}\right)$$
$$=\ \hbar^n\cdot\sum\nolimits_{s=1}^{n}\left(F_\hbar{}^{\vee}\right)^{\widehat{\otimes}\,(s-1)}\widehat{\otimes}\,\mathcal{I}_\hbar^{\curlyvee}\,\widehat{\otimes}\left(F_\hbar{}^{\vee}\right)^{\widehat{\otimes}\,(n-s)}$$

for all $n\in\mathbb{N}_+$, which means exactly that $\left(\mathcal{I}_\hbar^{\curlyvee}\right)^!\supseteq\mathcal{I}_\hbar$. Similarly, we have also $\delta_n\big(\mathcal{C}_\hbar\big)\subseteq J_{F_\hbar}^{\widetilde{\otimes}\,(n-1)}\,\widetilde{\otimes}\,\mathcal{C}_\hbar\ \subseteq\ \left(\hbar^{n-1}\left(F_\hbar{}^{\vee}\right)^{\widehat{\otimes}\,(n-1)}\right)\widehat{\otimes}\left(\hbar\,\mathcal{C}_\hbar^{\triangledown}\right)=\hbar^n\cdot\left(F_\hbar{}^{\vee}\right)^{\widehat{\otimes}\,(n-1)}\widehat{\otimes}\,\mathcal{C}_\hbar^{\triangledown}$ for all $n\in\mathbb{N}_+$, which means exactly that $\left(\mathcal{C}_\hbar^{\triangledown}\right)^{\Lsh}\supseteq\mathcal{C}_\hbar$. On the other hand, by definitions $\mathfrak{I}_\hbar^{\,!}\cap J_{F_\hbar}=\varepsilon\big(\mathfrak{I}_\hbar^{\,!}\cap J_{F_\hbar}\big)+\delta_1\big(\mathfrak{I}_\hbar^{\,!}\cap J_{F_\hbar}\big)=\delta_1\big(\mathfrak{I}_\hbar^{\,!}\cap J_{F_\hbar}\big)\subseteq\hbar\,\big(\mathfrak{I}_\hbar\cap J_{F_\hbar}\big)$, which implies $\left(\mathfrak{I}_\hbar^{\,!}\right)^{\curlyvee}\subseteq\mathfrak{I}_\hbar$. Similarly, $\mathfrak{C}_\hbar^{\Lsh}\cap J_{F_\hbar}=\varepsilon\big(\mathfrak{C}_\hbar^{\Lsh}\cap J_{F_\hbar}\big)+\delta_1\big(\mathfrak{C}_\hbar^{\Lsh}\cap J_{F_\hbar}\big)=\delta_1\big(\mathfrak{C}_\hbar^{\Lsh}\cap J_{F_\hbar}\big)\subseteq\hbar\cdot\big(\mathfrak{C}_\hbar\cap J_{F_\hbar}\big)$ yields $\left(\mathfrak{C}_\hbar^{\Lsh}\right)^{\triangledown}\subseteq\mathfrak{C}_\hbar$. Thus all identities in the claim are half proved.

To prove the reverse inclusions $\left(\mathcal{I}_\hbar^{\curlyvee}\right)^!\subseteq\mathcal{I}_\hbar$ and $\left(\mathcal{C}_\hbar^{\triangledown}\right)^{\Lsh}\subseteq\mathcal{C}_\hbar$ one can resume the proof of Proposition 3.2 in [Ga1], which shows that $\left(F_\hbar{}^{\vee}\right)'\subseteq F_\hbar$: in fact, the same arguments apply almost untouched with $\mathcal{C}_\hbar$ instead of $F_\hbar$, and also (with minimal changes) with $\mathcal{I}_\hbar$ instead of $F_\hbar$. The outcome is $\left(\mathcal{I}_\hbar^{\curlyvee}\right)^!\subseteq\mathcal{I}_\hbar$ and $\left(\mathcal{C}_\hbar^{\triangledown}\right)^{\Lsh}\subseteq\mathcal{C}_\hbar$, whence identities hold.

To finish with, by Proposition 4.4 we can apply twice Lemma 4.1 and get $\left(\mathfrak{C}_\hbar^{\Lsh}\right)^{\triangledown}=\left(\left(\mathfrak{I}_\hbar^{\,!}\right)^{\curlyvee}\right)^{\perp}$ and $\left(\mathfrak{I}_\hbar^{\,!}\right)^{\curlyvee}=\left(\left(\mathcal{C}_\hbar^{\triangledown}\right)^{\Lsh}\right)^{\perp}$. As $\left(\mathcal{I}_\hbar^{\curlyvee}\right)^!=\mathcal{I}_\hbar$ and $\left(\mathcal{C}_\hbar^{\triangledown}\right)^{\Lsh}=\mathcal{C}_\hbar$, we get $\left(\mathfrak{C}_\hbar^{\Lsh}\right)^{\triangledown}=\mathfrak{I}_\hbar{}^{\perp}$ and $\left(\mathfrak{I}_\hbar^{\,!}\right)^{\curlyvee}=\mathcal{C}_\hbar{}^{\perp}$; but then (2.4) eventually yields $\left(\mathfrak{C}_\hbar^{\Lsh}\right)^{\triangledown}=\mathfrak{C}_\hbar$ and $\left(\mathfrak{I}_\hbar^{\,!}\right)^{\curlyvee}=\mathfrak{I}_\hbar$. $\square$

Remark. Like for Lemma 4.2, in the previous proof we might prove only one of the identities in the claim, e.g. that for $\mathcal{I}_\hbar$: all others then follow via (i)–(iv) in Lemma 4.1.

5. Specialization at $\hbar = 0$

We shall now look at semiclassical limits of the images of Drinfeld-like functors. The result—Proposition 5.2—will be $\left(K^{\perp},\ G^*/K^{\perp}\right)$, in the sense that this will be the pair (*coisotropic subgroup*, *Poisson quotient*) mentioned in Proposition 4.4.

5.1. Lemma. *Let $\mathcal{S}(G^*)$ be the set of formal subgroups of the formal Poisson group G^*.*

(a) $\mathcal{I}_0{}^{\curlyvee}\trianglelefteq_\ell\dot{\trianglelefteq}\,F_0[[G]]^{\vee}=U(\mathfrak{g}^*)$, *whence* $\mathcal{I}_0{}^{\curlyvee}=U(\mathfrak{g}^*)\cdot\mathfrak{l}$ *for some Lie subalgebra* $\mathfrak{l}\leqslant\mathfrak{g}^*$;
(b) $\mathcal{C}_0^{\triangledown}\leq_{\mathcal{H}}F_0[[G]]^{\vee}=U(\mathfrak{g}^*)$, *whence* $\mathcal{C}_0^{\triangledown}=U(\mathfrak{h})$ *for some Lie subalgebra* $\mathfrak{h}\leqslant\mathfrak{g}^*$;
(c) $\mathfrak{I}_0{}^!\trianglelefteq_{\mathcal{H}}U_0(\mathfrak{g})'=F[[G^*]]$, *whence* $\mathfrak{I}_0^{\,!}=\mathcal{I}(\Gamma)$ *for some* $\Gamma\in\mathcal{S}(G^*)$;
(d) $\mathfrak{C}_0^{\Lsh}\leq^1\dot{\trianglelefteq}_\ell\,U_0(\mathfrak{g})'=F[[G^*]]$, *whence* $\mathfrak{C}_0^{\Lsh}=F[[G^*]]^{\theta}$ *for some* $\theta\in\mathcal{S}(G^*)$;
(e) *Let $H\in\mathcal{S}(G^*)$ be the formal subgroup of G^* with Lie$\,(H)=\mathfrak{h}$, and let $L\in\mathcal{S}(G^*)$ be the one with Lie$\,(L)=\mathfrak{l}$. Then $\Gamma=H=L=\theta$.*
(f) *the formal subgroup $\Gamma=H=L=\theta$ in* (e) *is coisotropic in G^*.*

Proof. Statements (a) and (d) follow trivially from Lemma 4.2; the same also implies part of (b) and (c), in that $\mathfrak{I}_0^!$ is a *bialgebra ideal* of $U_0(\mathfrak{g})'$ and $\mathcal{C}_0^\nabla$ is a *subbialgebra* of $F_0[[G]]^\vee$. Now, $F_0[[G]]^\vee = U(\mathfrak{g}^*)$, and a subbialgebra of any universal enveloping algebra (such as $U(\mathfrak{g}^*)$) is automatically a Hopf subalgebra: thus $\mathcal{C}_0^\nabla$ is a Hopf subalgebra. On the other hand, the orthogonality relations of Lemma 5.1(ii) imply that $\mathfrak{I}_0^!$ is a Hopf ideal too.

Claim (e) follows directly from Proposition 4.4 and from Remark 2.8(b).

Finally (f) follows from Proposition 4.4 and Lemma 2.9. □

5.2. Proposition. *The coisotropic subgroup* $\Gamma = H = L = \theta$ *of Proposition* 5.1 *coincide with* $K^\perp \in co\mathcal{S}(G^*)$ (*cf. Definition* 1.4). *In other words,* $\mathfrak{l} = \mathfrak{h}$ *coincides with* $\mathfrak{k}^\perp \big(\subseteq \mathfrak{g}^* \big)$. □

Proof. We resume the construction made for the proof of Lemma 4.3, with same notation. In particular we fix a special subset $\{j_1, \dots, j_k, j_{k+1}, \dots, j_n\}$ of J_G enjoying the properties mentioned there, and call $\{y_1, \dots, y_k, y_{k+1}, \dots, y_n\}$ its image in $\mathfrak{g}^* = J_G / J_G^{\,2}$.

The same kind of analysis carried on in the proof of Lemma 4.3 to prove that $\sigma \colon \mathfrak{g}^* \cong \mathfrak{t}$ shows that the unital subalgebra $\mathcal{C}_0^\nabla := \mathcal{C}_\hbar^\nabla \bmod \hbar\, F_\hbar^\vee$ is generated by $\eta\big(\mathcal{C}_\hbar^\nabla \cap J^\vee\big) = (\mu \circ \eta)\big(\mathcal{C}_\hbar \cap J\,\big) = (\sigma \circ \rho \circ \pi)\big(\mathcal{C}_\hbar \cap J\,\big) = \sigma\big(\rho(\mathcal{C} \cap J_G)\big) = \sigma\big(\rho(\langle j_{k+1}, \dots, j_n \rangle)\big) = \sigma\big(\mathfrak{k}^\perp\big)$, where $\langle\, j_{k+1}, \dots, j_n \rangle$ is the ideal of $\mathcal{C}$ generated by $\{j_{k+1}, \dots, j_n\}$. Therefore $\mathcal{C}_0^\nabla = U(\mathfrak{h})$ is generated by $\mathfrak{k}^\perp$, whose elements are primitive, so belong to $\mathfrak{h}$: then $\mathfrak{h} = \mathfrak{k}^\perp$. □

5.3. Corollary. $\mathcal{I}(K)_\hbar^\curlyvee$, $\mathcal{C}(K)_\hbar^\triangledown$, $\mathfrak{I}(K)_\hbar^!$ *and* $\mathfrak{C}(K)_\hbar^\urcorner$ *all provide quantizations, w.r.t.* $\big(U_\hbar{}', F_\hbar{}^\vee\big)$, *of the formal coisotropic subgroup* $K^\perp$ *and the formal Poisson quotient* $G^*/K^\perp$.

Proof. The claim follows from Proposition 4.4, Lemma 5.1 and Proposition 5.2. □

Patching together all previous results, we can finally prove Theorem 3.3:

Proof of Theorem 3.3. Corollary 5.3 proves that the functors in (a) are well-defined on objects, and it is trivially clear that they are inclusion-preserving, so they do are functors. Proposition 4.5 proves the rest of claim (a), in particular that these functors are in fact equivalences. In addition, Corollary 5.3 also proves claim (b). □

6. Example: the Stokes matrices as Poisson homogeneous SL_n^*-space

6.1. The Poisson homogeneous SL_n^*-space of Stokes matrices. Let $G = SL_n(\Bbbk)$ endowed with the standard Poisson–Lie structure. We denote by $\mathfrak{d}$ the Cartan subalgebra of diagonal matrices in $\mathfrak{sl}_n(\Bbbk)$. With $\mathfrak{b}_+$ (resp. $\mathfrak{b}_-$) we denote the Borel subalgebra of upper (resp. lower) triangular matrices in $\mathfrak{sl}_n$; then B_+ and B_- will be the corresponding Borel subgroups in SL_n. It is well known that at the infinitesimal level the dual Lie bialgebra can be identified with $\mathfrak{g}^* = \left\{(X, Y) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- \,\middle|\, X\big|_{\mathfrak{d}} = -Y\big|_{\mathfrak{d}}\right\}$, so that the simply connected dual Poisson group is $G^* = B_+ \star B_-$, the pairs of upper and lower triangular matrices such that the restrictions on the diagonal are mutually inverse.

By construction, the algebra $F[G^*] = F[B_+ \star B_-]$ is generated by matrix coefficients $x_{i,j}$ $(1 \leqslant i \leqslant j \leqslant n)$ for the over-diagonal part of B_+, $y_{i,j}$ $(1 \geqslant i \geqslant j \geqslant 1)$ for the under-diagonal part of B_-, and z_i $(1 \leqslant i \leqslant n)$ for the diagonal part of B_+.

Let $H = SO_n(\Bbbk) \hookrightarrow SL_n(\Bbbk)$ be the standard embedding. The corresponding Lie algebra is $\mathfrak{h} = \mathfrak{so}_n(\Bbbk)$. Its orthogonal in $\mathfrak{g}^*$, for the pairing given by the Killing form, is $\mathfrak{h}^\perp = \left\{\left(b, -b^t\right) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- \,:\, b\big|_{\mathfrak{d}} = 0\right\}$ and can be integrated to $H^\perp = \left\{(B, C) \in B_+ \star B_- \,\middle|\, BC^t = Id\right\}$, which is a coisotropic subgroup of G^*. We are then in the situation described in §1. The spaces SL_n/SO_n and $SL_n^*/H^\perp$ are a complementary dual pair of Poisson homogeneous spaces: the former can be identified with the space of symmetric matrices and the latter with the space U_n^+ of Stokes matrices, i.e. upper triangular unipotent $(n \times n)$-matrices. By construction the function algebra $F\left[U_n^+\right] = F\left[G^*/H^\perp\right] = F[G^*]^{H^\perp}$ is generated by elements $x_{i,j}$, for all $1 \leqslant i < j \leqslant n$, which may be realized as the matrix coefficient functions on Stokes matrices.

The Poisson structure on U_n^+ was first found by Dubrovin in the $n = 3$ case (see [Du]) and then by Ugaglia (cf. [Ug]) for generic $n \geqslant 3$ in a completely different setting: it naturally arises in the study of moduli spaces of semisimple Frobenius manifolds. Later, in [Bo,Xu], it was shown how U_n^+ with such structure is a Poisson homogeneous space of the Poisson–Lie group $B_+ \star B_-$, dual to the standard SL_n, as just explained. More explicitly, from [Xu] one can argue the following

6.2. Proposition. *Let $\Psi : B_+ \star B_- \longrightarrow B_+ \star B_-$, $\Psi(B, C) := \left(C^t, B^t\right)$ and let $H^\perp = \left\{ g \in B_+ \star B_- \,\middle|\, \psi(g) = g^{-1} \right\}$. Then $H^\perp$ is a coisotropic subgroup of $B_+ \star B_-$ and $U_n^+ \cong (B_+ \star B_-)/H^\perp$ with its quotient Poisson structure.*

6.3. Towards quantization of Stokes matrices. In the present section we look for quantizations of U_n^+: the first step is to switch to the associated formal homogeneous space. Actually, the function algebra $F_\hbar\left[\left[G^*/H^\perp\right]\right] = F_\hbar\left[\left[U_n^+\right]\right]$ is nothing but the algebra of formal power series in the matrix coefficient functions, say $\chi_{i,j}$ $(1 \leqslant i < j \leqslant n)$, on U_n^+.

Now we look for a quantization $F_\hbar\left[\left[U_n^+\right]\right]$ of $F\left[\left[U_n^+\right]\right]$ with the above Poisson structure: we shall find it applying Theorem 3.3. As our purpose is to obtain a quantum algebra of functions on the homogeneous space, an object of type (b) in the list (2.3), we start with an object of type (d) in the same list. This means that as a starting point we need a subalgebra and left coideal inside $U_\hbar(\mathfrak{sl}_n)$ quantizing the standard embedding of $\mathfrak{so}_n$. This has been already obtained in [No, §2.3] (see also the works of Klimyk et al., e.g [GIK] and references therein): we recall hereafter its definition in the formal setup. We begin fixing notation for $U_\hbar(\mathfrak{gl}_n)$, a quantum analogue of $U(\mathfrak{gl}_n)$, and its Hopf subalgebra $U_\hbar(\mathfrak{sl}_n)$:

6.4. Definition. We call $U_\hbar(\mathfrak{gl}_n)$ the topological, $\hbar$-adically complete, associative unital $\Bbbk[[\hbar]]$-algebra with generators f_i, ℓ_j, e_i $(i = 1, \dots, n-1;\ j = 1, \dots, n)$ and relations

$$\begin{aligned}
&\ell_j\, f_i - f_i\, \ell_j = (\delta_{i+1,j} - \delta_{i,j})\, f_i, \quad \ell_j\, \ell_k = \ell_k\, \ell_j, \\
&\quad \ell_j\, e_i - e_i\, \ell_j = (\delta_{i,j} - \delta_{i+1,j})\, e_i \quad \forall i, j, k, \\
&e_k\, f_l\, -\, f_l\, e_k = \delta_{k,l}\, \frac{t_k^{+1} - t_k^{-1}}{q - q^{-1}} \quad \forall k, l, \\
&e_i\, e_j = e_j\, e_i, \quad f_i\, f_j = f_j\, f_i \quad \forall |\, i - j\,| > 1, \\
&e_i^2\, e_j - \left(q + q^{-1}\right) e_i\, e_j\, e_i + e_j\, e_i^2 = 0, \\
&\quad f_i^2\, f_j - \left(q + q^{-1}\right) f_i\, f_j\, f_i + f_j\, f_i^2 = 0 \quad \forall |\, i - j\,| = 1,
\end{aligned}$$

where hereafter we use notation $q := \exp(\hbar)$, $q^X := \exp\left(\hbar\, X\right)$ and $t_i := q^{\ell_i - \ell_{i+1}}$ $(\forall\, i)$. It has a structure of topological Hopf $\Bbbk[[\hbar]]$-algebra uniquely given by

$$\begin{aligned}
&\Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad S(f_i) = -f_i\, t_i, \quad \varepsilon(f_i) = 0 \quad \forall i, \\
&\Delta(\ell_j) = \ell_j \otimes 1 + 1 \otimes \ell_j, \quad S(\ell_j) = -\ell_j, \quad \varepsilon(\ell_j) = 0 \quad \forall j, \\
&\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad S(e_i) = -t_i^{-1}\, e_i, \quad \varepsilon(e_i) = 0 \quad \forall i.
\end{aligned}$$

6.5. Quantum root vectors and L-operators in $U_\hbar(\mathfrak{gl}_n)$. We recall the notion of L-operators, first introduced in [FRT]: these are elements $L_{i,j}^{\pm} \in U_\hbar(\mathfrak{gl}_n)$ (with $i, j = 1, \dots, n$), which are defined as follows. Set $[x, y]_a := xy - ayx$ (for all x, y, a), and define

$$\begin{aligned}
&E_{i,i+1} := e_i, \quad E_{i,j} := \left[E_{i,k}, E_{k,j}\right]_q, \quad F_{i+1,i} := f_i, \\
&\quad F_{j,i} := \left[F_{j,k}, F_{k,i}\right]_{q^{-1}} \quad \forall\ i < k < j
\end{aligned}$$

(where $q := \exp(\hbar)$ again). These are *quantum root vectors* in $U_\hbar(\mathfrak{gl}_n)$, in that the coset of $E_{i,j}$ (resp. $F_{j,i}$) modulo $\hbar\, U_\hbar(\mathfrak{gl}_n)$ in $U_\hbar(\mathfrak{gl}_n) / \hbar\, U_\hbar(\mathfrak{gl}_n) \cong U(\mathfrak{gl}_n)$ is the elementary matrix $\mathrm{e}_{i,j}$ (resp. $\mathrm{e}_{j,i}$) for all $i < j$.

The L-operators are obtained by twisting and rescaling the above quantum root vectors,

$$\begin{aligned}
&L_{i,i}^{+} := q^{+\ell_i} =: g_i^{+1}, \quad L_{i,j}^{+} := +\left(q - q^{-1}\right) g_i^{+1} F_{j,i}, \quad L_{j,i}^{+} := 0 \quad (i < j), \\
&L_{i,i}^{-} := q^{-\ell_i} =: g_i^{-1}, \quad L_{i,j}^{-} := -\left(q - q^{-1}\right) E_{j,i}\, g_j^{-1}, \quad L_{j,i}^{-} := 0 \quad (i > j)
\end{aligned}$$

and satisfy the remarkable formulas $\Delta\big(L^{\pm}_{i,j}\big) = \sum_{k=i\wedge j}^{i\vee j} L^{\pm}_{(i\wedge j),k} \otimes L^{\pm}_{k,(i\vee j)}$, $\varepsilon\big(L^{\pm}_{i,j}\big) = \delta_{i,j}$.

When suitably normalized, the L-operators are again q-analogues of the elementary matrices of $\mathfrak{gl}_n$: namely, the coset of $\big(q-q^{-1}\big)^{-1} L^{+}_{i,j}$ (resp. $\big(q-q^{-1}\big)^{-1} L^{-}_{j,i}$) modulo $\hbar\, U_\hbar(\mathfrak{gl}_n)$ in the semiclassical limit $U_\hbar(\mathfrak{gl}_n)/\hbar\, U_\hbar(\mathfrak{gl}_n) \cong U(\mathfrak{gl}_n)$ is $\mathrm{e}_{j,i}$ (resp. $\mathrm{e}_{i,j}$) for all $i<j$. Moreover, the elements $\widehat{L}^{\pm}_{i,j} := \big(q-q^{-1}\big)^{\delta_{i,j}-1} L^{\pm}_{i,j}$ for $i \neq j$ together with the ℓ_k's form a set of generators for $U_\hbar(\mathfrak{gl}_n)$. Set also $\Lambda^{\pm} := \big(\Lambda^{\pm}_{i,j}\big)_{i,j=1}^{n}$ for any $\Lambda \in \big\{L, \widehat{L}\big\}$.

6.6. Quantization of $U(\mathfrak{sl}_n)$. For all $i = 1, \dots, n-1$, let $h_i := \ell_i - \ell_{i+1}$. Given $U_\hbar(\mathfrak{gl}_n)$ as above, we define $U_\hbar(\mathfrak{sl}_n)$ as the closed topological subalgebra of $U_\hbar(\mathfrak{gl}_n)$ generated by $\big\{f_i, h_i, e_i\big\}_{i=1,\dots,n-1}$. From the presentation of $U_\hbar(\mathfrak{gl}_n)$ in Definition 6.4 one argues a presentation of $U_\hbar(\mathfrak{sl}_n)$ as well: in particular, this shows that $U_\hbar(\mathfrak{sl}_n)$ is a *Hopf subalgebra* of $U_\hbar(\mathfrak{gl}_n)$; moreover, by construction we have a quantum analogue of the classical embedding $\mathfrak{sl}_n \hookrightarrow \mathfrak{gl}_n$. Note also that, for any i, j, we have $L^{\pm}_{i,j} \notin U_\hbar(\mathfrak{sl}_n)$. It is also immediate to check that our Hopf algebra $U_\hbar(\mathfrak{sl}_n)$ coincides with Drinfeld's one.

6.7. Quantization of $U(\mathfrak{so}_n)$. Following an idea of Noumi, Klimyk et al., we define $U_\hbar(\mathfrak{so}_n)$ as a subalgebra of $U_\hbar(\mathfrak{sl}_n)$. We call $U_\hbar(\mathfrak{so}_n)$ the closed topological $\Bbbk[[\hbar]]$-subalgebra of $U_\hbar(\mathfrak{gl}_n)$ generated by the matrix entries of $K := \big(\widehat{L}^{-}\big)^{t} J\, L^{+} = \big(L^{-}\big)^{t} J\, \widehat{L}^{+}$, where J is the $(n \times n)$ diagonal matrix $diag\big(q^{n-1}, \dots, q, 1\big)$. Explicit computations give

$$K_{i,j} = \sum_{k=i}^{j} q^{n-k} \big(q-q^{-1}\big)^{-1} L^{-}_{k,i}\, L^{+}_{k,j} = \sum_{k=i}^{j} q^{n-k} \widehat{L}^{-}_{k,i}\, L^{+}_{k,j} = \sum_{k=i}^{j} q^{n-k} L^{-}_{k,i}\, \widehat{L}^{+}_{k,j}$$

for the matrix entries of K, which is upper triangular with J onto the diagonal. Note that we have $\big(q-q^{-1}\big)^{\delta_{i,j}-1} L^{-}_{k,i}\, L^{+}_{k,j} \in U_\hbar(\mathfrak{sl}_n)$ for all i, k, j, hence $U_\hbar(\mathfrak{so}_n) \subseteq U_\hbar(\mathfrak{sl}_n)$ as well. This yields quantum analogues of the classical embeddings $\mathfrak{so}_n \hookrightarrow \mathfrak{sl}_n \hookrightarrow \mathfrak{gl}_n$. Moreover, w.r.t. the Lie bialgebra structure on $\mathfrak{g}$ inherited by its quantization $U_\hbar(\mathfrak{gl}_n)$ one has that $\mathfrak{so}_n$ is also a Lie coideal of $\mathfrak{gl}_n$, hence correspondingly SO_n is a coisotropic subgroup of GL_n. Note that we have fixed Noumi's parameters a_j to be $a_j = q^{n-j}$ (for all j). With respect to the coproduct, $U_\hbar(\mathfrak{so}_n)$ *is a right coideal* both of $U_\hbar(\mathfrak{sl}_n)$ and of $U_\hbar(\mathfrak{gl}_n)$. Thus $\mathfrak{C}_\hbar := U_\hbar(\mathfrak{so}_n)$ and $U_\hbar(\mathfrak{g}) := U_\hbar(\mathfrak{sl}_n)$ do realize the situation of (2.3(d))—the specialization result $U_\hbar(\mathfrak{so}_n)\big|_{\hbar=0} \cong U(\mathfrak{so}_n)$ being explained in [No]—but for having a *right* instead than *left* coideal. However, *by left–right symmetry our analysis remains unchanged*. So $\mathfrak{C}_\hbar := U_\hbar(\mathfrak{so}_n)$ is a quantum subgroup for the quantum group $U_\hbar(\mathfrak{gl}_n)$.

We now apply the functor $(\ )^{\urcorner} : \mathfrak{C}_\hbar\big(co\mathcal{S}(SL)\big) \xrightarrow{\cong} \mathcal{C}_\hbar\big(co\mathcal{S}(B_+ \star B_-)\big)$ of Theorem 3.3 to get a quantization $F_\hbar\big[\big[U_n^{+}\big]\big] := U_\hbar(\mathfrak{so}_n)^{\urcorner}$ of $F\big[\big[SO_n^{\perp}\big]\big] = F\big[\big[U_n^{+}\big]\big]$. We explain in detail the case of $n = 3$, and then basing on that we will give a sketch of the general situation. Note that the over-diagonal entries of the matrix K will provide—passing from $U_\hbar(\mathfrak{so}_n)$ to $F_\hbar\big[\big[U_n^+\big]\big] := U_\hbar(\mathfrak{so}_n)^{\curlyvee}$ and eventually to the semiclassical limit of the latter—algebra generators of $F\big[\big[U_n^+\big]\big]$, namely the matrix coefficients of Stokes matrices.

Warning: Noumi's definition of $U_\hbar(\mathfrak{so}_n)$ is in [No, §2.4] (mutatis mutandis). It is explained there that one can take as algebra generators of $U_\hbar(\mathfrak{so}_n)$ the entries of either one of four different matrices, given in formula (2.18) in [*loc. cit.*]. Among these, we choose $K_0 := \big(L^-\big)^t\, Q\, J^{-1}\, L^+$, where J is given above and Q is the $(n \times n)$ diagonal matrix $diag\big(q^{n-1}, \dots, q, 1\big) = J^2$, so that $Q\, J^{-1} = J$. We also need to rescale such generators, and eventually take $K := \big(q - q^{-1}\big)^{-1} K_0$ as above for the purpose of specialization.

6.8. The algebras $U_\hbar(\mathfrak{gl}_n)'$ and $U_\hbar(\mathfrak{sl}_n)'$. As $F_\hbar\big[\big[U_n^+\big]\big] := U_\hbar(\mathfrak{so}_n)^{\curlyvee}$ is a subalgebra of $U_\hbar(\mathfrak{gl}_n)'$ and $U_\hbar(\mathfrak{sl}_n)'$, we do need a clear description of these objects.

By definition, the topological Hopf algebra $U_\hbar(\mathfrak{gl}_n)$ is *Q-graded*, Q being the root lattice of $\mathfrak{gl}_n$, with $\partial(f_i) = -\alpha_i$, $\partial(h_i) = 0$, $\partial(e_i) := +\alpha_i$ where α_i is the ith simple root of $\mathfrak{gl}_n$, for all i. Also, $\partial\big(F_{j,i}\big) = \partial\big(\Lambda^+_{i,j}\big) = -\sum_{i \leqslant k \leqslant j} \alpha_k =: -\alpha_{i,j}$ and $\partial\big(E_{i,j}\big) = \partial\big(\Lambda^-_{j,i}\big) = +\sum_{i \leqslant k \leqslant j} \alpha_k =: +\alpha_{i,j}$, for all $i < j$ and $\Lambda \in \big\{L, \widehat{L}\big\}$. It follows that $U_\hbar(\mathfrak{gl}_n)^{\otimes d}$ is $Q^{\oplus d}$-graded as a topological algebra, and the like for $U_\hbar(\mathfrak{sl}_n)^{\otimes d}$ (for all $d \in \mathbb{N}$).

The formulas for the coproduct of L-operators in §6.5 can be iterated, yielding for $\widehat{L}^\pm$

$$\Delta^d\big(\widehat{L}^+_{i,j}\big) = \sum\nolimits_{I_d^+} \big(q - q^{-1}\big)^{(d-1-\delta_{i,k_1}-\delta_{k_1,k_2}-\cdots-\delta_{k_{d-1},j})}\, \widehat{L}^+_{i,k_1} \otimes \widehat{L}^+_{k_1,k_2} \otimes \cdots \otimes \widehat{L}^+_{k_{d-1},j},$$

where $I_d^+ := \big\{\, k_1, \dots, k_{dl-1} \,\big|\, i \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_{d-1} \leqslant j \big\}$ for $i < j$, and similarly

$$\Delta^d\big(\widehat{L}^-_{i,j}\big) = \sum\nolimits_{I_d^-} \big(q - q^{-1}\big)^{(d-1-\delta_{i,k_1}-\delta_{k_1,k_2}-\cdots-\delta_{k_{d-1},j})}\, \widehat{L}^-_{i,k_1} \otimes \widehat{L}^-_{k_1,k_2} \otimes \cdots \otimes \widehat{L}^-_{k_{d-1},j},$$

where $I_d^- := \big\{\, k_1, \dots, k_{d-1} \,\big|\, i \geqslant k_1 \geqslant k_2 \geqslant \cdots \geqslant k_{d-1} \geqslant j \,\big\}$ for $i > j$. In particular,

$$\Delta^d\big(\widehat{L}^\varepsilon_{i,j}\big) = \sum\nolimits_{r+s=\ell-1} \big(g_i^{\varepsilon 1}\big)^{\otimes r} \otimes \widehat{L}^\varepsilon_{i,j} \otimes \big(g_j^{\varepsilon 1}\big)^{\otimes s} + R \quad (\text{hereafter } \ \varepsilon \in \{+,-\}),$$

where R is a topological sum of homogeneous terms in $U_\hbar(\mathfrak{gl}_n)^{\otimes \ell}$ whose degree in $Q^{\oplus d}$ is of type $(\partial_1, \dots, \partial_d)$, each ∂_k being a positive or negative root (according to $\varepsilon = -$ or $\varepsilon = +$) of height less than that of $\alpha_{i,j}$. Finally, for all $i = 1, \dots, n$ we have

$$\Delta^d\big(h_i\big) = \sum\nolimits_{r+s=d-1} 1^{\otimes r} \otimes h_i \otimes 1^{\otimes s} \quad \forall d \in \mathbb{N}_+.$$

Now let Φ_+ (resp. Φ_-) be the set of positive (resp. negative) roots of $\mathfrak{gl}_n$, and fix any total ordering $\preceq$ on Φ_+. Set also $L^\pm_\alpha := L^\pm_{i,j}$ for each root $\alpha = \mp\alpha_{i,j}$. The well-known quantum PBW theorem (adapted to the present case) ensures that

$$\mathbb{S} := \left\{ \prod_{\alpha\in\Phi_-} \big(\widehat{L}_\alpha^+\big)^{\Lambda_\alpha^+} \prod_{i=1}^n \ell^{\,\eta_i} \prod_{\alpha\in\Phi_+} \big(\widehat{L}_\alpha^-\big)^{\Lambda_\alpha^-} \;\middle|\; \Lambda_\alpha^+, \eta_i, \Lambda_\alpha^+ \in \mathbb{N} \;\forall\; \alpha, i \right\}$$

is a topological $\Bbbk[[\hbar]]$-basis of $U_\hbar(\mathfrak{gl}_n)$; hereafter the products over positive or negative roots are made w.r.t. the fixed total ordering.

Given $\mathcal{M}\in\mathbb{S}$ we set $\big|\mathcal{M}\big| := \sum_{\alpha\in\Phi_-} \Lambda_\alpha^+ + \sum_{i=1}^n \eta_i + \sum_{\alpha\in\Phi_+} \Lambda_\alpha^-$, the sum of all exponents occurring in $\mathcal{M}$. Since Δ^d is a *graded algebra morphism*, the previous formulas imply that for each PBW-like monomial $\mathcal{M}$ in $\mathbb{S}$ we have, for all $d \geqslant \big|\mathcal{M}\big|$,

$$\begin{aligned}
\Delta^d\big(\mathcal{M}\big) = {} & \widehat{L}^+_{-\alpha_1}\zeta^{(1)}_{-\alpha_1} \otimes \cdots \otimes \widehat{L}^+_{-\alpha_1}\zeta^{(\lambda^+_{-\alpha_1})}_{-\alpha_1} \otimes \cdots \otimes \widehat{L}^+_{-\alpha_N}\zeta^{(1)}_{-\alpha_N} \otimes \cdots \otimes \widehat{L}^+_{-\alpha_N}\zeta^{(\lambda^+_{-\alpha_N})}_{-\alpha_N} \\
& \otimes h_1\,\theta^{(1)}_{h_1} \otimes \cdots \otimes h_1\theta^{(\eta_1)}_{h_1} \otimes \cdots \otimes h_{n-1}\theta^{(1)}_{h_{n-1}} \otimes \cdots \otimes h_{n-1}\theta^{(\eta_{n-1})}_{h_{n-1}} \\
& \otimes \widehat{L}^-_{+\alpha_1}\zeta^{(1)}_{+\alpha_1} \otimes \cdots \otimes \widehat{L}^-_{+\alpha_1}\zeta^{(\lambda^+_{+\alpha_1})}_{+\alpha_1} \otimes \cdots \otimes \widehat{L}^-_{+\alpha_N}\zeta^{(1)}_{+\alpha_N} \otimes \cdots \otimes \widehat{L}^-_{+\alpha_N}\zeta^{(\lambda^+_{+\alpha_N})}_{+\alpha_N} \\
& \otimes \psi_1 \otimes \cdots \otimes \psi_{d-|\mathcal{M}|} + T,
\end{aligned}$$

where $\alpha_1 \preceq \alpha_2 \preceq \cdots \preceq \alpha_N$ (with $N = \binom{n}{2}$) are the positive roots of $\mathfrak{gl}_n$, each one of the $\zeta^{(k)}_{-\alpha_r}$'s, the $\theta^{(s)}_{h_i}$'s, the $\zeta^{(\ell)}_{+\alpha_r}$'s and the ψ_p's is a suitable monomial in the $g_j^{\pm 1}$'s, and finally T is a sum of homogeneous terms whose degrees are different from the degree of the previous summand. From this and $\varepsilon\big(\widehat{L}^\pm_{i,j}\big) = 0 = \varepsilon(\ell_k)$ (for all k and all $i \neq j$) we argue

$$\begin{aligned}
\delta_d\big(\mathcal{M}\big) = {} & \widehat{L}^+_{-\alpha_1}\zeta^{(1)}_{-\alpha_1} \otimes \cdots \otimes \widehat{L}^+_{-\alpha_1}\zeta^{(\lambda^+_{-\alpha_1})}_{-\alpha_1} \otimes \cdots \otimes \widehat{L}^+_{-\alpha_N}\zeta^{(1)}_{-\alpha_N} \otimes \cdots \otimes \widehat{L}^+_{-\alpha_N}\zeta^{(\lambda^+_{-\alpha_N})}_{-\alpha_N} \\
& \otimes h_1\theta^{(1)}_{h_1} \otimes \cdots \otimes h_1\theta^{(\eta_1)}_{h_1} \otimes \cdots \otimes h_{n-1}\theta^{(1)}_{h_{n-1}} \otimes \cdots \otimes h_{n-1}\theta^{(\eta_{n-1})}_{h_{n-1}} \\
& \otimes \widehat{L}^-_{+\alpha_1}\zeta^{(1)}_{+\alpha_1} \otimes \cdots \otimes \widehat{L}^-_{+\alpha_1}\zeta^{(\lambda^+_{+\alpha_1})}_{+\alpha_1} \otimes \cdots \otimes \widehat{L}^-_{+\alpha_N}\zeta^{(1)}_{+\alpha_N} \otimes \cdots \otimes \widehat{L}^-_{+\alpha_N}\zeta^{(\lambda^+_{+\alpha_N})}_{+\alpha_N} \\
& \otimes (\psi_1 - 1) \otimes \cdots \otimes (\psi_{d-|\mathcal{M}|} - 1) + P,
\end{aligned} \tag{6.1}$$

where $P := (id - \varepsilon)^{\otimes d}\big(T\big)$ is again a sum of homogeneous terms whose degrees are different from that of the previous summand (which is homogeneous too). In the latter each tensor factor belongs to $U_\hbar(\mathfrak{gl}_n) \setminus \hbar U_\hbar(\mathfrak{gl}_n)$, whilst $(\psi_k - 1) \in \hbar U_\hbar(\mathfrak{gl}_n) \setminus \hbar^2 U_\hbar(\mathfrak{gl}_n)$ for all k: the outcome is $\delta_d\big(\mathcal{M}\big) \in \hbar^{d-|\mathcal{M}|} U_\hbar(\mathfrak{gl}_n) \setminus \hbar^{d-|\mathcal{M}|+1} U_\hbar(\mathfrak{gl}_n)$ for all $d \geqslant \big|\mathcal{M}\big|$, whence

$$\widetilde{\mathcal{M}} := \hbar^{|\mathcal{M}|}\,\mathcal{M} \in U_\hbar(\mathfrak{gl}_n)' \setminus \hbar\, U_\hbar(\mathfrak{gl}_n)' \quad \forall \mathcal{M} \in \mathbb{S}.$$

From this we eventually get $\widetilde{\mathbb{S}} := \left\{ \widetilde{\mathcal{M}} \;\middle|\; \mathcal{M} \in \mathbb{S} \right\} \subseteq U_\hbar(\mathfrak{gl}_n)'$, thus also the $\Bbbk[[\hbar]]$-span of $\widetilde{\mathbb{S}}$ is contained in $U_\hbar(\mathfrak{gl}_n)'$. In fact, the previous analysis also allows to revert this last result, thus proving the following

Claim. $\widetilde{\mathbb{S}}$ *is a topological* $\Bbbk[[\hbar]]$*-basis of* $U_\hbar(\mathfrak{gl}_n)'$.

Indeed, let $\eta \in U_\hbar(\mathfrak{gl}_n)'$ and take an expansion $\eta = \sum_{\mathcal{M}\in\mathbb{S}} c_\mathcal{M}\, \mathcal{M}$ of η of minimal length as a linear combination over $\Bbbk[[\hbar]]$ of elements of $\mathbb{S}$. Let us call $\mathcal{M}^{d,\otimes}$ the first summand in right-hand side of (6.1): then our analysis gives

$$\delta_d(\eta)=\sum_{\mathcal{M}\in\mathbb{S}} c_\mathcal{M}\, \delta_d\big(\mathcal{M}\big)=\sum_{\mathcal{M}\in\mathbb{S}} c_\mathcal{M}\, \Big(\mathcal{M}^{d,\otimes}+P\Big)=\sum_{|\mathcal{M}|=\mu_+} c_\mathcal{M}\, \mathcal{M}^{d,\otimes} + R_-,$$

where $\mu_+ := \max\big\{|\mathcal{M}|\big\}_{\mathcal{M}\in\mathbb{S}}$ and R_- is a sum of homogeneous terms whose degrees are different from the degrees of any summand in $\sum_{|\mathcal{M}|=\mu_+} c_\mathcal{M}\, \mathcal{M}^{d,\otimes}$. Therefore $\delta_d(\eta) \in \hbar^d\, U_\hbar(\mathfrak{gl}_n)$ (as $\eta \in U_\hbar(\mathfrak{gl}_n)'$) forces also $\sum_{|\mathcal{M}|=\mu_+} c_\mathcal{M}\, \mathcal{M}^{d,\otimes} \in \hbar^d\, U_\hbar(\mathfrak{gl}_n)$. Again by a simple degree argument we get $\sum_{\substack{|\mathcal{M}|=\mu_+\\ \partial(\mathcal{M})=\beta}} c_\mathcal{M}\, \mathcal{M}^{d,\otimes} \in \hbar^d\, U_\hbar(\mathfrak{gl}_n)$ for all $\beta \in Q$. Using linear independence of monomials in the $\widehat{L}^{\pm}_{\alpha}$'s with different exponents (consequence of the quantum PBW theorem) we get also $\sum_{\mathcal{M}\in S_{\mu_+}} c_\mathcal{M}\, \mathcal{M}^{d,\otimes} \in \hbar^d\, U_\hbar(\mathfrak{gl}_n)$ where S_{μ_+} is the set of all monomials $\mathcal{M}$ with $|\mathcal{M}| = \mu_+$ and *fixed* exponents $\lambda^{\pm}_{\alpha}$. Again by quantum PBW, this happens if and only if $\sum_{\mathcal{M}\in S_{\mu_+}} c_\mathcal{M}\, \mathcal{M} \in \hbar^{\mu_+}\, U_\hbar(\mathfrak{gl}_n)$, which in turn implies $c_\mathcal{M} \in \hbar^{\mu_+}\, \Bbbk[[\hbar]]$ for all $\mathcal{M}$ involved; so this last sum can be written as $\eta_+ = \sum_{\mathcal{M}\in S_{\mu_+}} c_\mathcal{M}\, \mathcal{M} = \sum_{\mathcal{M}\in S_{\mu_+}} \widehat{c}_\mathcal{M}\, \widetilde{\mathcal{M}}$, which belongs to the topological $\Bbbk[[\hbar]]$-span of $\widetilde{\mathbb{S}}$, with $\widehat{c}_\mathcal{M} := \hbar^{-\mu_+} c_\mathcal{M} \in \Bbbk[[\hbar]]$. But then also $\eta' := \eta - \eta_+ \in U_\hbar(\mathfrak{gl}_n)'$, and η' has less non-zero coefficients in its expansion w.r.t. the topological $\Bbbk[[\hbar]]$-basis $\mathbb{S}$. Iterating this argument, we eventually find that η belongs to the topological $\Bbbk[[\hbar]]$-span of $\widetilde{\mathbb{S}}$.

Note that each $\widetilde{M} \in \widetilde{\mathbb{S}}$ is a monomial in the elements $\widetilde{\ell}_k := \hbar\, \ell_k$ and the $\hbar\, \widehat{L}^{\pm}_{\mp\alpha} = \hbar, \big(q-q^{-1}\big)^{-1} L^{\pm}_{\mp\alpha}$, hence *these are topological algebra generators for* $U_\hbar(\mathfrak{gl}_n)'$. Furthermore, since $\hbar\big(q-q^{-1}\big)^{-1}$ is an invertible element of $\Bbbk[[\hbar]]$, we have also that $U_\hbar(\mathfrak{gl}_n)'$ *is generated, as a unital* $\Bbbk[[\hbar]]$*-algebra, by the* $L^{\pm}_{i,j}$*'s and the* $\widetilde{\ell}_k$*'s (for all* i, j, k*).*

In the semiclassical limit $U_\hbar(\mathfrak{gl}_n)'\Big|_{\hbar=0} \cong F\big[\big[GL^*\big]\big] = F\big[\big[B^G_+ \star B^G_-\big]\big] = F\big[\big[\mathfrak{b}^G_+ \star \mathfrak{b}^G_-\big]\big]$, these generators specialize to matrix coefficients onto $\mathfrak{b}^G_+ \star \mathfrak{b}^G_-$; hereafter $B^G_\pm$ is the Borel subgroup in GL_n of upper/lower triangular matrices and, $\mathfrak{b}^G_\pm := Lie\big(B^G_\pm\big)$, so $B^G_+ \star B^G_-$ is the Poisson group dual to GL^*_n, and we identify $B^G_+ \star B^G_- \cong \mathfrak{b}^G_+ \star \mathfrak{b}^G_-$ (everything is very similar to the case of SL_n). Namely, for every $i<j$ the coset modulo $\hbar\, U_\hbar(\mathfrak{gl}_n)'$ of each $L^{+}_{i,j}$ is the matrix coefficient $e_{i,j}$ onto $\big(\mathfrak{b}^G_+, \mathbf{0}\big) \cong \mathfrak{b}^G_+$, and the coset of each $L^{-}_{j,i}$ is the matrix coefficient $e_{j,i}$ onto $\big(\mathbf{0}, \mathfrak{b}^G_-\big) \cong \mathfrak{b}^G_-$; also, for each k the coset of $\widetilde{\ell}_k$ modulo $\hbar\, U_\hbar(\mathfrak{gl}_n)'$ is $e_{k,k}\Big|_{B^G_+} = e^{-1}_{k,k}\Big|_{B^G_-}$. Finally, as $L^{\pm}_{k,k} =: g^{\pm 1}_k := \exp\big(\hbar\, \ell_k\big) = \exp\big(\widetilde{\ell}_k\big)$ the same kind of relation occurs between the cosets modulo $\hbar\, U_\hbar(\mathfrak{gl}_n)'$ of $L^{\pm}_{k,k}$ and of $\widetilde{\ell}_k$, for all k.

As for $U_\hbar(\mathfrak{sl}_n)'$, for all $i<j$ we have that $\widetilde{F}_{j,i} := \left(q-q^{-1}\right) F_{j,i} = g_i^{-1}\, L_{i,j}^{+}$ and $\widetilde{E}_{i,j} := -\left(q-q^{-1}\right) E_{i,j} = -L_{j,i}^{-}\, g_i^{+1}$ belong to $U_\hbar(\mathfrak{gl}_n)' \bigcap U_\hbar(\mathfrak{sl}_n) = U_\hbar(\mathfrak{sl}_n)'$, as well as $\widetilde{h}_k := \hbar\,(\ell_k - \ell_{k+1}) = \widetilde{\ell}_k - \widetilde{\ell}_{k+1}$ (for all k). Indeed, with the same analysis as above—up to the obvious, minimal changes—one proves also that $U_\hbar(\mathfrak{sl}_n)'$ *is generated, as a topological unital* $\Bbbk[[\hbar]]$*-algebra, by the* $\widetilde{F}_{j,i}$*'s, the* $\widetilde{E}_{i,j}$*'s (for all* $i<j$*) and the* $\widetilde{h}_k$*'s (for all* k*).*

In addition, $U_\hbar(\mathfrak{sl}_n)'$ has $\Bbbk[[\hbar]]$-basis the set of rescaled PBW-like monomials (in the above generators) analogue to the set $\widetilde{\mathbb{S}}$ considered above which is a basis for $U_\hbar(\mathfrak{gl}_n)'$.

Finally, under specialization $U_\hbar(\mathfrak{sl}_n)'\Big|_{\hbar=0} \cong F\big[\big[\mathfrak{sl}_n^*\big]\big] = F\big[\big[B_+ \star B_-\big]\big] = F\big[\big[\mathfrak{b}_+ \star \mathfrak{b}_-\big]\big]$ the above generators specialize as $\widetilde{F}_{j,i}\big|_{\hbar=0} = e_{i,i}^{-1}\, e_{i,j}\big|_{\mathfrak{b}_+}$, $\widetilde{E}_{i,j}\big|_{\hbar=0} = e_{j,i}\; e_{i,i}^{+1}\big|_{\mathfrak{b}_-}$ (for all $i<j$) and $\widetilde{h}_k\big|_{\hbar=0} = e_{k,k}\big|_{\mathfrak{b}_+} - e_{k+1,k+1}\big|_{\mathfrak{b}_+}$ (for all $k=1,\dots,n-1$).

6.9. Quantum Stokes matrices: $n=3$. According to the general recipe in §6.7, the generators of $\mathcal{H} = U_\hbar(\mathfrak{so}_3)$ are

$$\begin{aligned} K_{1,2} &= q^2\left(F_1 - q\,T_1^{-1}E_1\right), \quad K_{2,3} = q\left(F_2 - q\,T_2^{-1}E_2\right),\\ K_{1,3} &= q^2\left(F_{3,1} - \left(q-q^{-1}\right)F_2\,T_1^{-1}E_1 - T_1^{-1}T_2^{-1}E_{1,3}\right), \end{aligned}$$

(cf. §6.7) where $T_s^{\pm 1} := t_s^{\pm 1}$ $(s=1,2)$. From this one can directly prove that

$$\big[K_{1,2}, K_{2,3}\big]_q = -\,q^2\,K_{1,3}. \tag{6.2}$$

Using the relations between the elements θ_j in [No, §2.4]—namely, formulas (2.23) therein—and remarking that $K_{1,2} = q\,\theta_1$, $K_{2,3} = \theta_2$, one can derive also

$$\big[K_{1,3}, K_{1,2}\big]_q = -\,q^3\,K_{2,3}, \quad \big[K_{2,3}, K_{1,3}\big]_q = -\,q\,K_{1,2}. \tag{6.3}$$

Indeed, the case $n=3$ is especially interesting because, using renormalized generators $\widetilde{K}_{1,2} := q^{-5/2}K_{1,2}$, $\widetilde{K}_{1,3} := q^{-4/2}K_{1,3}$ and $\widetilde{K}_{2,3} := q^{-3/2}K_{2,3}$ one has for $U_\hbar(\mathfrak{so}_3)$ a cyclically invariant presentation (see [HKP] and references therein, and Remark 6.11(b) too). However, this special feature has no general counterpart for $n \neq 3$.

The following PBW-like theorem holds for $U_\hbar(\mathfrak{so}_3)$, as a direct consequence of definitions and formulas (6.2–6.3):

Claim. *$U_\hbar(\mathfrak{so}_3)$ is a topologically free $\Bbbk[[\hbar]]$-module, with topological $\Bbbk[[\hbar]]$-basis the set of ordered monomials $\left\{K_{1,2}^{a}\,K_{1,3}^{b}\,K_{2,3}^{c}\,\middle|\,a,b,c\in\mathbb{N}\right\}$. A similar basis is the one with $\widetilde{K}_{i,j}$ instead of $K_{i,j}$ everywhere.*

6.10. Theorem. *$F_\hbar\big[\big[U_3^+\big]\big] := U_\hbar(\mathfrak{so}_3)^{\urcorner}$ is the topological, $\hbar$-adically complete, unital $\Bbbk[[\hbar]]$-algebra with generators*

$$k_{1,2} := q^{-2}\left(q-q^{-1}\right)K_{1,2}, \quad k_{2,3} := q^{-1}\left(q-q^{-1}\right)K_{2,3},$$
$$k_{1,3} := q^{-2}\left(q-q^{-1}\right)K_{1,3}$$

and relations

$$\begin{aligned} k_{1,2}\,k_{2,3} &= q\,k_{2,3}\,k_{1,2} - q\left(q-q^{-1}\right)k_{1,3}, \\ k_{2,3}\,k_{1,3} &= q\,k_{1,3}\,k_{2,3} - \left(q-q^{-1}\right)k_{1,2}, \\ k_{1,3}\,k_{1,2} &= q\,k_{1,2}\,k_{1,3} - \left(q-q^{-1}\right)k_{2,3} \end{aligned} \tag{6.4}$$

with the right coideal structure given by

$$\begin{aligned} \Delta\big(k_{1,2}\big) &= 1\otimes k_{1,2} + k_{1,2}\otimes t_1^{-1}, \quad \Delta\big(k_{2,3}\big) = 1\otimes k_{2,3} + k_{2,3}\otimes t_2^{-1}, \\ \Delta\big(k_{1,3}\big) &= 1\otimes k_{1,3} + k_{1,3}\otimes t_1^{-1}t_2^{-1} + \left(q-q^{-1}\right)k_{1,2}\otimes f_2\,t_1^{-1} \\ &\quad -q^{-1}\left(q-q^{-1}\right)k_{2,3}\otimes t_1^{-1}t_2^{-1}e_1. \end{aligned}$$

Moreover, $F_\hbar\big[\big[U_3^+\big]\big] := U_\hbar(\mathfrak{so}_3)^{\urcorner}$ is a free $\Bbbk[[\hbar]]$-module, a $\Bbbk[[\hbar]]$-basis being the set of ordered monomials $\mathbb{B}_3 := \big\{k_{1,2}^a\,k_{1,3}^b\,k_{2,3}^c \,\big|\, a,b,c\in\mathbb{N}\big\}$.

Proof. The relations (6.4) among the $k_{i,j}$'s clearly spring out of formulas (6.2)–(6.3), whilst the formulas for the right coideal structure directly come out of the very definitions. The key point of the proof instead is to show that these elements do generate $U_\hbar(\mathfrak{so}_3)^{\urcorner}$.

From the above formulas for Δ, a straightforward computation proves that ($\forall d\in\mathbb{N}$)

$$\begin{aligned} \delta_d\big(k_{1,2}\big) &= k_{1,2}\otimes\left(t_1^{-1}-1\right)^{\otimes(d-1)}, \quad \delta_d\big(k_{2,3}\big) = k_{2,3}\otimes\left(t_2^{-1}-1\right)^{\otimes(d-1)}, \\ \delta_d\big(k_{1,3}\big) &= k_{1,3}\otimes\left(t_1^{-1}t_2^{-1}-1\right)^{\otimes(d-1)} \\ &\quad +\textstyle\sum_{r+s=d-2}\left(q-q^{-1}\right)k_{1,2}\otimes\left(t_1^{-1}-1\right)^{\otimes r}\otimes f_2\,t_1^{-1}\otimes\left(t_1^{-1}t_2^{-1}-1\right)^{\otimes s} \\ &\quad +\textstyle\sum_{r+s=d-2}q^{-1}\left(q-q^{-1}\right)k_{2,3} \\ &\quad \otimes\left(t_2^{-1}-1\right)^{\otimes r}\otimes t_1^{-1}t_2^{-1}e_1\otimes\left(t_1^{-1}t_2^{-1}-1\right)^{\otimes s}. \end{aligned}$$

As $k_{i,j}$, $\left(t_h^{-1}-1\right), \left(t_1^{-1}t_2^{-1}-1\right)\in\hbar\,U_\hbar(\mathfrak{so}_3)\setminus\hbar^2\,U_\hbar(\mathfrak{so}_3)$, we have $k_{1,2},k_{2,3},k_{1,3}\in U_\hbar(\mathfrak{so}_3)^{\urcorner}\setminus\hbar\,U_\hbar(\mathfrak{so}_3)^{\urcorner}$, so the subalgebra generated by these elements lies in $U_\hbar(\mathfrak{so}_3)^{\urcorner}$.

We shall now prove that $\mathbb{B}_3$ is a topological $\Bbbk[[\hbar]]$-basis of $U_\hbar(\mathfrak{so}_3)^{\urcorner}$; this in turn will imply that this algebra is generated by $k_{1,2}$, $k_{2,3}$ and $k_{1,3}$. First, the claim in §6.9

implies that $\mathbb{B}_3$ is a linearly independent set inside $U_\hbar(\mathfrak{so}_3)^{\daleth}$; then now we prove that it spans $U_\hbar(\mathfrak{so}_3)^{\daleth}$ over $\Bbbk[[\hbar]]$. The formulas for Δ on the $k_{i,j}$'s give also, for all $d \in \mathbb{N}$,

$$\begin{aligned}
\Delta^d\big(K_{1,2}\big) &= \sum_{r+s=d-1} 1^{\otimes r} \otimes K_{1,2} \otimes \big(t_1^{-1}\big)^{\otimes s}, \\
\Delta^d\big(K_{2,3}\big) &= \sum_{r+s=d-1} 1^{\otimes r} \otimes K_{2,3} \otimes \big(t_2^{-1}\big)^{\otimes s} \\
\Delta^d\big(K_{1,3}\big) &= \sum_{r+s=d-1} 1^{\otimes r} \otimes K_{1,3} \otimes \big(t_1^{-1} t_2^{-1}\big)^{\otimes s} \\
&\quad + \sum_{r+p+s=d-2} 1^{\otimes r} \otimes K_{1,2} \otimes \big(t_1^{-1}\big)^{\otimes p} \otimes A \otimes \big(t_1^{-1} t_2^{-1}\big)^{\otimes s} \\
&\quad + \sum_{r+p+s=d-2} 1^{\otimes r} \otimes K_{2,3} \otimes \big(t_2^{-1}\big)^{\otimes p} \otimes B \otimes \big(t_1^{-1} t_2^{-1}\big)^{\otimes s}
\end{aligned}$$

with $A := L_{2,3}^{+}\, g_1^{-1} = \big(q - q^{-1}\big) f_2\, t_1^{-1}$, $B := q^{-1} g_3\, L_{2,1}^{-} = -\,q^{-1}\big(q - q^{-1}\big)\, t_1^{-1} t_2^{-1} e_1 \in U_\hbar(\mathfrak{sl}_n)'$. In particular, this implies that $\delta_{a+2b+c}\big(K_{1,2}^{\,a}\, K_{1,3}^{\,b}\, K_{2,3}^{\,c}\big) = \sum_{i \in I} C_{i,1} \otimes C_{i,2} \otimes \cdots \otimes C_{i,a+2b+c}$ (for some index set I) where each tensor factor $C_{i,j}$ is a product of type

$$C_{i,j} = t_1^{-n_1} t_2^{-\nu_1} \cdot D_1 \cdot t_1^{-n_2} t_2^{-\nu_2} \cdot D_2 \cdot \cdots \cdot t_1^{-n_{k-1}} t_2^{-\nu_{k-1}} \cdot D_{k-1} \cdot t_1^{-n_k} t_2^{-\nu_k} \qquad (k \in \mathbb{N}_+)$$

with $n_s, \nu_s \in \mathbb{N}$ and $D_s \in \big\{K_{1,2},\, K_{1,3},\, K_{2,3},\, A,\, B\,\big\} \bigcup \big\{\big(t_1^{-\tau_1} t_2^{-\tau_2} - 1\big) \,\big|\, \tau_1, \tau_2 \in \mathbb{N}_+\big\}$. In particular—cf. also (6.4)—there is a first summand of type

$$\begin{aligned}
\Phi_1^{a,b,c} = {}& \left(\bigotimes_{p=0}^{a-1} t_1^{-p} K_{1,2}\right) \otimes \left(\bigotimes_{r=0}^{b-1} t_1^{-(a+r)} t_2^{-r}\right) \otimes \left(\bigotimes_{s=0}^{c-1} t_1^{-(a+b)} t_2^{-(b+s)} K_{2,3}\right) \\
& \otimes \big(t_1^{-(a+b)} t_2^{-(b+c-1)} - 1\big)^{\otimes b}.
\end{aligned}$$

Define the *length* of $K_{1,2}^{\,a}\, K_{1,3}^{\,b}\, K_{2,3}^{\,c} \in \mathbb{B}_3$ as $l\big(K_{1,2}^{\,a}\, K_{1,3}^{\,b}\, K_{2,3}^{\,c}\big) := a + 2b + c$, and let $\mathcal{H}_n$ be the $\Bbbk[[\hbar]]$-span of all monomials in $\mathbb{B}_3$ of length at most n. This defines an algebra filtration $\{\mathcal{H}_n\}_{n \in \mathbb{N}}$ of $U_\hbar(\mathfrak{so}_3)$; the formulas for the coproduct of the $k_{i,j}$'s show that this is a *comodule algebra filtration*, i.e. an algebra filtration such that $\Delta\big(\mathcal{H}_n\big) \subseteq \mathcal{H}_n \otimes U_\hbar(\mathfrak{sl}_3)$ for all n. A similar filtration is also induced onto each tensor power $U_\hbar(\mathfrak{so}_3)^{\otimes l}$ ($l \in \mathbb{N}$).

Any $\eta \in U_\hbar(\mathfrak{so}_3)^{\daleth}$ expands uniquely as $\eta = \sum_{a,b,c \in \mathbb{N}} \chi_{a,b,c}\, K_{1,2}^{\,a}\, K_{1,3}^{\,b}\, K_{2,3}^{\,c}$ for some $\chi_{a,b,c} \in \Bbbk[[\hbar]]$, by the claim of §6.9. Set $\mu := \min\big\{a + 2b + c \,\big|\, \chi_{a,b,c} \neq 0\big\}$, and look at $\delta_\mu(\eta) = \sum_{a,b,c \in \mathbb{N}} \chi_{a,b,c} \cdot \delta_\mu\big(K_{1,2}^{\,a}\, K_{1,3}^{\,b}\, K_{2,3}^{\,c}\big) \in \hbar^{\,\mu}\, U_\hbar(\mathfrak{so}_3) \otimes U_\hbar(\mathfrak{sl}_3)^{\otimes(\mu-1)}$. By degree arguments—w.r.t. the filtration $\{\mathcal{H}_n\}_{n \in \mathbb{N}}$ of $U_\hbar(\mathfrak{so}_3)$ given above—we see that

$\delta_\mu(\eta) \in \hbar^{\,\mu}\, U_\hbar(\mathfrak{so}_3) \otimes U_\hbar(\mathfrak{sl}_3)^{\otimes(\mu-1)}$ forces also

$$\sum_{a+2\,b+c=\mu} \chi_{a,b,c} \cdot \delta_\mu\big(K_{1,2}^{\,a}\, K_{1,3}^{\,b}\, K_{2,3}^{\,c}\big) \in \hbar^{\,\mu}\, U_\hbar(\mathfrak{so}_3) \otimes U_\hbar(\mathfrak{sl}_3)^{\otimes(\mu-1)}. \tag{6.5}$$

By the analysis above, each $\delta_\mu\big(K_{1,2}^{\,a}\, K_{1,3}^{\,b}\, K_{2,3}^{\,c}\big)$ in (6.5) is equal to $\Phi_1^{a,b,c}$ (defined above) plus other terms which are linearly independent of $\Phi^{a,b,c}$ modulo $\hbar\, U_\hbar(\mathfrak{sl}_n)^{\otimes\mu}$. Furthermore, all these $\Phi_1^{a,b,c}$'s, for different triples $(a,b,c)\in\mathbb{N}^3$, are linearly independent inside $U_\hbar(\mathfrak{sl}_n)^{\otimes\mu}$, by construction. As an outcome, we have that (6.5) implies

$$\chi_{a,b,c} \cdot \Phi_1^{a,b,c} \in \hbar^{\,\mu}\, U_\hbar(\mathfrak{so}_3) \otimes U_\hbar(\mathfrak{sl}_3)^{\otimes(\mu-1)} \quad \forall\ a+2\,b+c=\mu.$$

Since $\Phi_1^{a,b,c} \in \hbar^b\, U_\hbar(\mathfrak{so}_3) \otimes U_\hbar(\mathfrak{sl}_3)^{\otimes(\mu-1)}$ by construction, we argue $\chi_{a,b,c} \in \hbar^{a+b+c}\Bbbk[[\hbar]]$ for all $a+2\,b+c=\mu$, so that

$$\chi_{a,b,c}\, K_{1,2}^{\,a}\, K_{1,3}^{\,b}\, K_{2,3}^{\,c} \in \Bbbk[[\hbar]] \cdot k_{1,2}^{\,a}\, k_{1,3}^{\,b}\, k_{2,3}^{\,c} \subseteq \Bbbk[[\hbar]]\text{-span of } \mathbb{B}_3 \quad \forall\ a+2\,b+c=\mu.$$

But then $\eta_- := \sum_{a+2\,b+c=\mu} \chi_{a,b,c} \cdot K_{1,2}^{\,a}\, K_{1,3}^{\,b}\, K_{2,3}^{\,c} \in U_\hbar(\mathfrak{so}_3)^{\urcorner}$ by our previous results, hence also

$$\eta_> := \eta - \eta_- = \sum_{a+2\,b+c>\mu} \chi_{a,b,c} \cdot K_{1,2}^{\,a}\, K_{1,3}^{\,b}\, K_{2,3}^{\,c} \in U_\hbar(\mathfrak{so}_3)^{\urcorner}.$$

Now we can apply the same arguments to $\eta_<$ instead of η: iterating this procedure (involving monomials in the $K_{i,j}$'s whose length grows up), we eventually find that η belongs to the topological $\Bbbk[[\hbar]]$-span of $\mathbb{B}_3$. $\square$

6.11. Remarks. (a) In §6.8 we saw that $U_\hbar(\mathfrak{sl}_n)'$ is generated by the L-operators, hence its semiclassical limit $F[[G^*]]$ is generated by their cosets, which are simply half the matrix coefficients generating $F[[G^*]]$ (see §6.1). Then by the very construction and our concrete description of $U_\hbar(\mathfrak{so}_3)^{\urcorner}$ we get that the generators $k_{i,j}$ specialize, in $U_\hbar(\mathfrak{so}_3)^{\urcorner}\Big|_{\hbar=0} = F\big[\big[U_3^+\big]\big]$, right to the generators of $F\big[\big[U_3^+\big]\big]$ (cf. §6.1). In particular, the corresponding limit Poisson bracket can therefore be verified to be equal to that in [Ug] and in [Xu] (the latter taken from [Du]), up to normalizations: e.g., the isomorphism between our presentation of $F\big[\big[U_3^+\big]\big]$ and Xu's one is given by

$$k_{1,2}\big|_{\hbar=0} \mapsto z, \quad k_{1,3}\big|_{\hbar=0} \mapsto y, \quad k_{2,3}\big|_{\hbar=0} \mapsto x$$

(notation of [Xu], §1, formula (2)), and this is easily seen to preserve the Poisson bracket.

(b) The claim and proof of Theorem 6.10 show that one could take as generators for $U_\hbar(\mathfrak{so}_3)^{\urcorner}$ simply the $\big(q-q^{-1}\big)\, K_{i,j}$'s. However, our choice of normalization (dividing out such generators by suitable powers of q) lead us to better looking relations, such as (6.4). Indeed, this can still be improved, taking new generators $\widetilde{k}_{1,2} := q^{-1/2}\, k_{1,2} = \big(q - q^{-1}\big)\widetilde{K}_{1,2}$, $\widetilde{k}_{1,3} := k_{1,3} = \big(q - q^{-1}\big)\widetilde{K}_{1,3}$ and $\widetilde{k}_{2,3} := q^{-1/2}\, k_{2,3} = \big(q - q^{-1}\big)\widetilde{K}_{2,3}$ (see §6.9): these enjoy the relations $\widetilde{k}_{1,2}\,\widetilde{k}_{2,3} = q\,\widetilde{k}_{2,3}\,\widetilde{k}_{1,2} - \big(q - q^{-1}\big)\,\widetilde{k}_{1,3}$, $\widetilde{k}_{2,3}\,\widetilde{k}_{1,3} = q\,\widetilde{k}_{1,3}\,\widetilde{k}_{2,3} - \big(q - q^{-1}\big)\,\widetilde{k}_{1,2}$, $\widetilde{k}_{1,3}\,\widetilde{k}_{1,2} = q\,\widetilde{k}_{1,2}\,\widetilde{k}_{1,3} - \big(q - q^{-1}\big)\,\widetilde{k}_{2,3}$, which are *totally symmetric with respect to cyclic permutations of the indices*. Nevertheless, this special feature—like for $U_\hbar(\mathfrak{so}_3)$—has no general counterpart for $n \neq 3$.

6.12. The general case. Let us now move to the general case $n > 3$. The generators $K_{i,j}$ $(i < j)$ are defined in §6.7; like in the claim in §6.9, we have a PBW-like theorem for $U_\hbar(\mathfrak{so}_n)$: namely, the set of all ordered monomials (w.r.t. any fixed total order of the set of pairs $\big\{(i,j)\,\big|\, i < j\,\big\}$) in the $K_{i,j}$'s is a topological $\Bbbk[[\hbar]]$-basis of $U_\hbar(\mathfrak{so}_n)$.

Straightforward computations yield

$$\delta_d\big(K_{i,j}\big) = \sum_I\; K_{t_1,s_1} \otimes (id - \varepsilon)\big(L^-_{t_1,t_2} L^+_{s_1,s_2}\big) \otimes \cdots \otimes (id - \varepsilon)\big(L^-_{t_{d-2},i}\, L^+_{s_{d-2},j}\big),$$

where the set of indices is $I = \big\{\, i \leqslant t_{d-2} \leqslant \cdots \leqslant t_1 < s_1 \leqslant \cdots \leqslant s_{d-2} \leqslant j\,\big\}$; it is worth pointing out that, while the L-operators $L^+_{i,j}$ and $L^-_{i,j}$ do not belong to $U_\hbar(\mathfrak{sl}_n)$ but only to $U_\hbar(\mathfrak{gl}_n)$, the products $L^-_{t_r,t_{r+1}} L^+_{s_r,s_{r+1}}$ do belong to $U_\hbar(\mathfrak{sl}_n)$. From this one gets easily

$$\delta_d\big(K_{i,j}\big) \in \hbar^{d-1} U_\hbar(\mathfrak{so}_n) \otimes U_\hbar(\mathfrak{sl}_n)^{\otimes(d-1)} \quad (i < j,\, d \in \mathbb{N}),$$

whence $k_{i,j} := \big(q - q^{-1}\big)\, K_{i,j} \in U_\hbar(\mathfrak{so}_n)^{\curlyvee} \setminus \hbar\, U_\hbar(\mathfrak{so}_n)^{\curlyvee}$ follows at once.

Indeed, with much the same analysis as in §§6.9–10 one can prove that in fact the $k_{i,j}$'s (for $i<j$) form a complete set of generators for the algebra $U_\hbar(\mathfrak{so}_n)^{\curlyvee}$, and that the set of ordered monomials in these generators is a topological $\Bbbk[[\hbar]]$-basis for $U_\hbar(\mathfrak{so}_n)^{\curlyvee}$. Finding the relations between the $k_{i,j}$'s then will provide an explicit presentation of the algebra $U_\hbar(\mathfrak{so}_n)^{\curlyvee}$, hence a quantization $F_\hbar\big[\big[U_n^+\big]\big] := U_\hbar(\mathfrak{so}_n)^{\curlyvee}$ of $F\big[\big[U_n^+\big]\big]$ with the Poisson structure given in [Ug], the analogue of Remark 6.11(a) holding true in the general case too.

7. Generalizations

7.1. Quantum duality with half quantizations. In the present work we take from scratch the datum of a pair of mutually dual quantum groups, namely $\big(F_\hbar[[G]],\, U_\hbar(\mathfrak{g})\big)$ (cf. §2.7). In the proofs, this assumption is exploited to apply orthogonality arguments, for which all these are necessary (a single quantum groups would not be enough).

However, this is only a matter of choice. Indeed, our quantum duality principle deals with quantum subgroups which are contained either in $F_\hbar[[G]]$ or in $U_\hbar(\mathfrak{g})$, and *we* *might prove every step in our discussion using only the single quantum group which is concerned*, and only one quantum subgroup (such as $\mathcal{I}_\hbar$, or $\mathcal{C}_\hbar$, etc.) at the time, by a direct method which use no orthogonality arguments. To give a sample, we re-prove part of Lemma 4.2:

Claim. *Let* ${\mathcal{I}_\hbar}^{\curlyvee}$ *and* ${\mathcal{C}_\hbar}^{\triangledown}$ *be as in Lemma* 4.2. *Then* ${\mathcal{I}_\hbar}^{\curlyvee} \trianglelefteq^{\cdot} F_\hbar[[G]]^\vee$ *and* ${\mathcal{C}_\hbar}^{\triangledown} \dot{\leq} F_\hbar[[G]]^\vee$.

Proof. By definition ${\mathcal{I}_\hbar}^{\curlyvee}$ is the left ideal of $F_\hbar[[G]]^\vee$ generated by $\hbar^{-1}\mathcal{I}_\hbar$, hence it is enough to show that $\Delta\big(F_\hbar[[G]]^\vee \cdot \hbar^{-1}\mathcal{I}_\hbar\big) \subseteq F_\hbar[[G]]^\vee \otimes {\mathcal{I}_\hbar}^{\curlyvee} + {\mathcal{I}_\hbar}^{\curlyvee} \otimes F_\hbar[[G]]^\vee$. Since $\mathcal{I}_\hbar$ is a coideal of $F_\hbar[[G]]$ (see §2.6), we have $\Delta\big(F_\hbar[[G]]^\vee \cdot \hbar^{-1}\mathcal{I}_\hbar\big) \subseteq \big(F_\hbar[[G]]^\vee \otimes F_\hbar[[G]]^\vee\big) \cdot \big(F_\hbar[[G]] \otimes \hbar^{-1}\mathcal{I}_\hbar + \hbar^{-1}\mathcal{I}_\hbar \otimes F_\hbar[[G]]\big) \subseteq F_\hbar[[G]]^\vee \otimes {\mathcal{I}_\hbar}^{\curlyvee} + {\mathcal{I}_\hbar}^{\curlyvee} \otimes F_\hbar[[G]]^\vee$.

The case of ${\mathcal{C}_\hbar}^{\triangledown}$ is entirely similar. □

7.2. Quantum duality with global quantizations. In this paper, we use quantum groups in the sense of Definition 2.2; in literature, these are sometimes called *local* quantizations. Instead, one can consider *global quantizations*: quantum groups like Jimbo's, Lusztig's, etc. The latter ones differ from the former in two respects:

(1) they are standard (rather than topological) Hopf algebras;
(2) they may be defined over any ring R, the rôle of $\hbar$ being played by a suitable element of that ring (the most common example is $R = \Bbbk\big[q, q^{-1}\big]$ and $\hbar = q - 1$).

The first point implies that the semiclassical limit of a quantum group of this type is either $U(\mathfrak{g})$, for some Lie bialgebra $\mathfrak{g}$, or $F[G]$, the algebra of regular functions on some Poisson algebraic group G. The latter is a geometrical object of *global* type, thus a quantum group specializing to it carries richer information than a QFSHA. The second point implies that one can consider different specializations, namely one for each point of the spectrum of the ground ring R: so this setting is richer from an arithmetical viewpoint.

Now, the present work might be written equally well in terms of *global quantum groups* and their specializations. The only care is to start with algebraic Poisson groups and algebraic Poisson homogeneous spaces, instead of formal ones. Then one defines Drinfeld-like functors in a perfectly similar manner; the key fact is that the quantum duality principle has a *global version* (see [Ga2]) in which the recipe given in §3 to define Drinfeld-like functors do make sense, up to a few technical details, in the global framework as well. In addition, one can also extend our quantum duality principle for coisotropic subgroups (and Poisson quotients) to all closed subgroups (and all homogeneous spaces): the outcome then is that applying the so-extended Drinfeld's functors to any closed subgroup (or homogeneous space) one always gets a *coisotropic* subgroup (or a Poisson quotient) of the dual Poisson group, and this is again characterized in terms of involutivity (see [CG]).

7.3. $*$-Structures and quantum duality for real subgroups and homogeneous spaces. If one is interested in quantizations of real subgroups and real homogeneous spaces, then $*$-structures must be considered on the quantum group Hopf algebras one starts from. It is then possible to perform all our construction in this setting, and to formulate and prove a version of the QDP for *real* quantum subgroups and quantum homogeneous spaces too, both in the formal and in the global setting; see [CG] for details.

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